FIBONACCI

HIS NUMBERS AND HIS RABBITS

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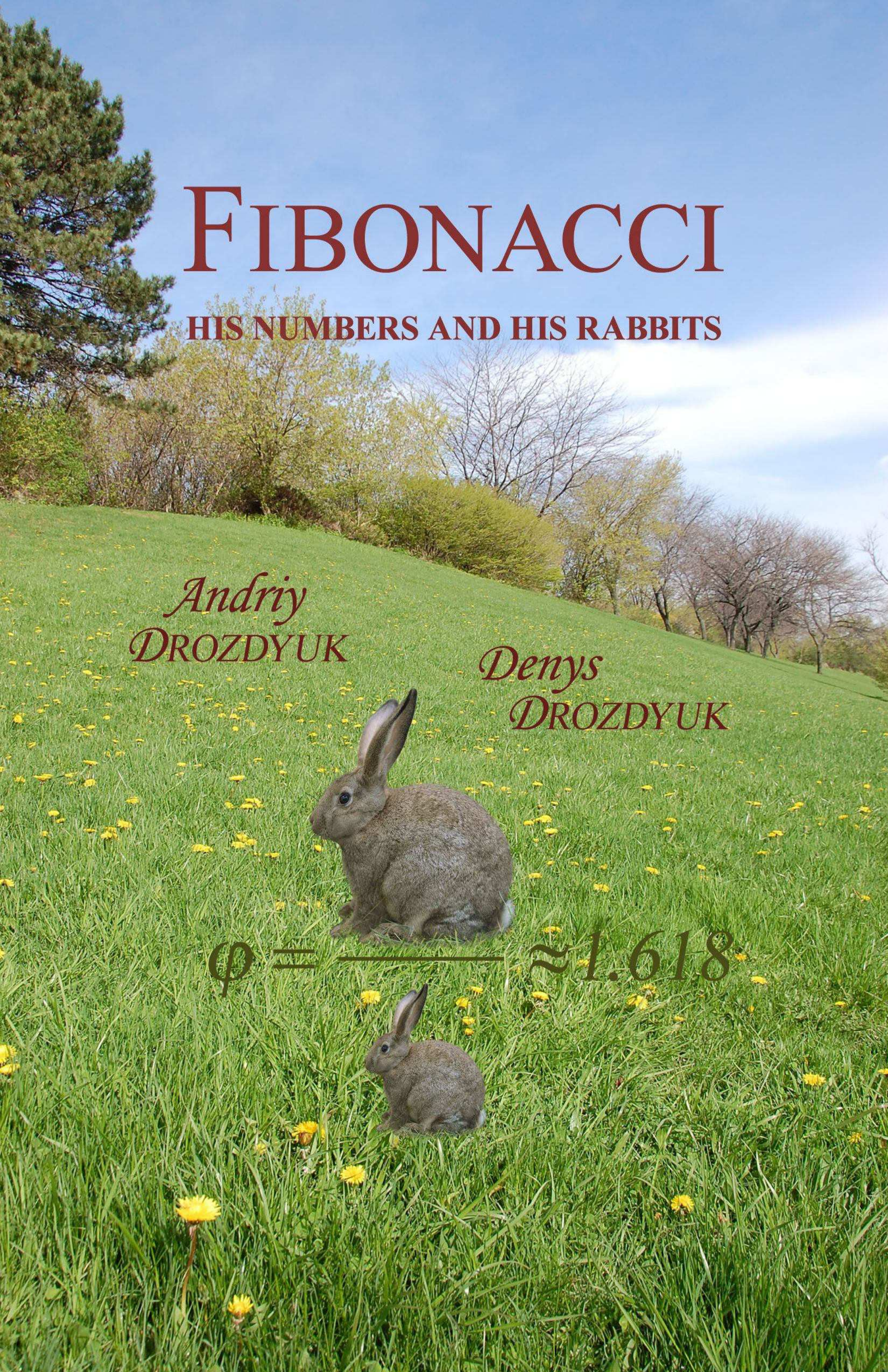

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This book contains what is known today about Fibonacci. The basic concepts about the Fibonacci numbers are given. The role of Fibonacci in the development of mathematics is considered and the value of Fibonacci's work "Liber Abaci" in the spread of Arabic numerals in Europe is assessed.

The book is intended for a wide range of readers.

It may be useful to students, teachers and scientists as it contains excerpts and quotes, taken directly from the original works (primary sources) and presented both in the original language and in the translation.

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to our mom and dad

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FOREWORD

There is a great deal written about the medieval mathematician Leonardo Fibonacci, despite an almost complete lack of evidence from his time. Due to lack of information, one can find discrepancies and sometimes contradictory opinions or statements about the biographical facts of Fibonacci's life and about the assessment of the importance of Fibonacci and his works for the development of mathematics.

Since — when referring to Fibonacci — not everyone will verify the validity of one or another existing statement about him or his works, a reference to a randomly chosen source can lead to replication of inaccurate information.

In addition, information about Fibonacci is scattered across numerous publications, some of which the reader may not be aware of or may find access to difficult.

Accordingly, we have tried to compile in one place — in this small book — everything known today about Fibonacci, using as primary sources only the original works, so the reader can form their own opinion on the subject matter based on the information provided by us. Quotations and excerpts from the original sources, including those of Fibonacci's works, are provided in the original language as well as in the translation.

It took us four years to write this book. A lot of time was spent searching, gathering and processing information.

Not all questions were answered. In such cases the problem is outlined and the attention of the reader is drawn to it.

It is our hope that this book will be useful to those just starting to get acquainted with Fibonacci, as well as to those, who already know quite a bit about him.

ACKNOWLEDGEMENTS

First of all we would like to say sincere words of gratitude to our parents for their help, support and constant attention to the writing of our work — to see it to the end.

This attention was a kind of moral incentive for us. We would like to thank our mother, Myroslava Kumka, as well as volunteers from the Internet for helping us with the translation from Latin. To our dad, Valery Drozdyuk, we are indebted for the illustrations he created and the routine work done by him. It was he who moved us to write this book by calling our attention to contradictory information about Fibonacci.

By visiting the Internet we were able to find out which questions related to Fibonacci and his works were raised and discussed, which opinions and judgments were voiced and which of them were interpreted ambiguously, differently or were contradictory to each other. Many of the rare original works, which we could not do without, can be found on the Internet today.

It is safe to say that without the Internet and the help of our parents this book would not be possible.

We would like to express our deepest appreciation to Professor Marion McGregor, PhD (Public Policy and Political Economy) and Director of Education (Canadian Memorial Chiropractic College), for carefully reading our book and for giving us valuable suggestions.

It is necessary to thank professor Jens Høyrup from Denmark for responding to us and answering some of our questions, as well as for uploading Fibonacci's works to the Internet.

Many thanks to Heinz Lueneburg for his participation in the mathematics forum on the Internet on the subject of Fibonacci, and for his considerate and attentive attitude toward us when we turned to him for advice, which he never refused or delayed in providing. Unfortunately, he was not able to evaluate our work — as it turned out, we took too long to write it.

Andriy Drozdyuk
Denys Drozdyuk
Toronto, 2010

Chapter I

FIBONACCI

In our everyday life we mention perhaps only two of the mathematicians that ever lived: *Pythagoras* in association with his famous theorem, and *Fibonacci* in association with his famous numbers. The Fibonacci numbers have many fascinating mathematical properties and can be found in the world, surrounding us, in the art and sciences.

Fibonacci's biography: conjectures only

Leonardo Fibonacci was the most prominent European mathematician of the Middle Ages. He was born in Pisa, Italy presumably in 1170 and died some time after 1240.

The times of Fibonacci were the times of the Crusades and of the establishment and rapid growth of the universities in Europe, the times of strong political conflicts between the Emperor of the Holy Roman Empire Frederick II (1194−1250) and the Papacy. The Italian maritime city-states of Pisa, Genoa, Venice and Amalfi were locked in intense trade rivalry throughout the Mediterranean world, including Byzantium and the Muslim countries [44; p. 3]. Countless travelers passed back and forth between Europe and the Middle East, and not a few adventurous and enterprising spirits dared to penetrate as far as India and China [39; p. 1]. Europe had emerged from the period of barbarian invasions and disruptions known as the Dark Ages, however there was a new great Mongol Empire growing in the East, and by the end of Fibonacci's life, defeating Russia, the Mongols entered Czech, Hungary and Poland, but it was not fated for them to go on or hold their position.

Little is known about Fibonacci's life. The only autobiographical fragment can be found at the very beginning of his manuscript "Liber Abaci." There also exists a document from the city of Pisa in which an annual salary is awarded to Fibonacci and a notarial act of purchase and sale (contract of sale), where Fibonacci and his relatives are mentioned. That is all. There are no eyewitness accounts, no descriptions, no portraits — nothing! Biographies were rare in the Middle Ages — even for kings and popes, portraits were even rarer. So all the portraits and statues of Fibonacci are nothing more than a figment of the imagination.

Professor Heinz Lueneburg (Heinz Lüneburg; 1935−2009) notes in the mathematics forum on the Internet that the basis for reckoning dates in Leonardo's biography is the date of his first version of *Liber Abaci* and the date of the *Pisan document* [34; 1) Re: Abbaci or Abaci?]:

> "Leonardo wrote the first version of his liber abbaci in 1202 (Pisan calendar). He had travelled a lot in advance. Hence he can't have been much younger than 30 when he wrote the book. So assume that he was born around 1170. Then he was about 71, when the Pisan document was written. So he can't have been much older than 30 when he wrote the first version of his liber abbaci. Hence: Leonardo Pisano was born around 1170 and died after 1240."

Pisan document

The Pisan document is a decree of the city-state of Pisa, awarding Leonardo yearly salary in recognition of his services to the city. The date of this decree according to different sources is: 1240, 1241, 1242.

The confusion arises because of the Pisan calendar. Their counting of the years is one year ahead of Florentine counting. Both towns had March 25th as the first day of the year; so the Pisan year 1242, Florence would count as the year 1241. This was changed under the Medici, when Pisa was under Florentine rule.

The changes to the Pisan system of counting the years did not occur only once, and the changes were not always made to all the documents, which hindered the work of later experts.

These peculiarities of the Pisan date reckoning system are noted by Baldassarre Boncompagni[1] in his work "Della vita e delle opere di Leonardo Pisano matematico del secolo decimoterzo" (*On the Life and Works of Leonardo Pisano, mathematician of the 13th Century*) after providing the record by notary Perizolo where he mentions Fibonacci and his work *Liber Abaci* [5; pp. 84−85].

Records by the notary of the Roman Empire Perizolo da Pisa (*Perizolo da Pisa, Notaro Imperiale*) is a chronology of some facts of the period 1422−1510. Perizolo in his records repeatedly remarks that he dates recorded by him events in accordance with the Pisan system of counting the years, which is why Boncompagni comments it.

Perizolo's records, titled "Ricordi di Ser Perizolo da Pisa, dall'anno 1422 sino al 1510," are in the second part of the sixth volume of the 1845 journal "Archivio Storico Italiano" [42; tomo VI, parte II, sezione II, pp. 385−396]. Volume VI is composed of two parts under the common title "Delle Istorie Pisane Libri XVI" (*The History of Pisa in 16 Books*) and includes a collection of some documents and 10 of the 16 books from the work "The History of Pisa" by the historian and erudite of 16th century canonical archpriest (*canonico arciprete*) Raffaello Roncioni from Pisa (around 1550−1619 [6; p. 30]). All of this is published under the editorship and with comments of professor Francesko Bonaini[2].

Lueneburg in his book "Leonardi Pisani Liber Abbaci oder Lesevergnügen eines Mathematikers" (*Leonardi Pisani's Liber*

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[1] Baldassarre Boncompagni Ludovisi, principe di Piombino (1821−1894) — Baldassarre Boncompagni Prince of Piombino of a rich and noble Roman family, mathematician by education, the Italian historian of mathematics and physics. He was the first, who at his own expenses published in print the works of Fibonacci.
[2] Francesco Bonaini (1806−1874) was an Italian historian, expert, erudite and archivist.

Abbaci or pleasure reading of a mathematician) also notes these peculiarities of the Pisan system of counting the years [35; p. 23].

Concerning the date of the Pisan document Lueneburg says the following [34; 2) Re: Bio of Fibonacci]:

> "The Pisan document is dated into 1241 by Bonaini who first described it. He does not give any reason for the dating. If one checks the original, as I did, being not an expert in old fashioned handwriting nor in the local history of Pisa there are other dating possibilies, too. The document itself is just a footnote to the Constitutum usus pisanae civitatis the dates of the alterations of which are kept in the end of the document not saying what was changed at the particular date."

About his discovery of the document, which he dates as 1241, Bonaini notifies the public in his announcement "Memoria unica sincrona di Leonardo Fibonacci novamente[3] scoperta" (*The unique record connected with Leonardo Fibonacci — new finding*) published in the historic journal "Giornale Storico" in 1857 [4].

In the announcement the text of the Pisan document and some additional information about Fibonacci are provided [4; p. 241]. Everything is supplemented by short comments of Bonaini. This announcement with an added introduction appears in 1867 as a separate brochure under the title "Iscrizione collocata nell'Archivio di Stato in Pisa a onore di Leonardo Fibonacci cui va unita una spiegazione del prof. Francesko Bonaini" (*The plate inscription in honor of Leonardo Fibonacci placed in the State Archive in Pisa together with an explanation of prof. Francesko Bonaini*) [3].

The Pisan document is usually dated 1241, while for the date of Fibonacci's death the year 1240 is usually used. Does this mean that Leonardo first died and then received the award? Of course not. Rather, this can be accounted for by approximations of the dates of events, relating to Fibonacci, and the transition to exact dates can lead to such inconsistencies.

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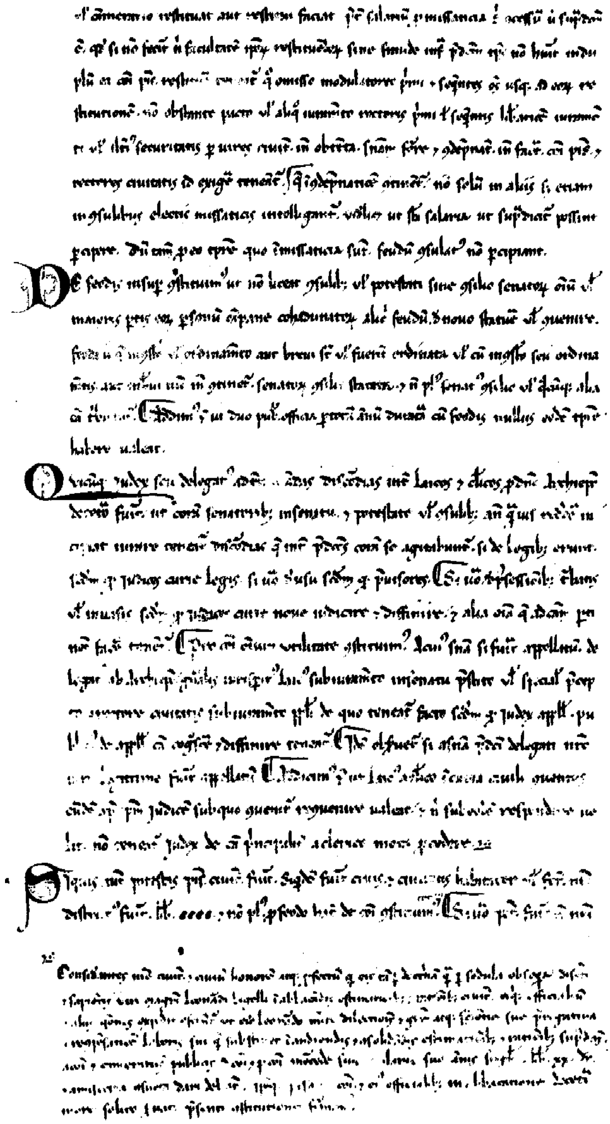
[3] Title of the article uses "novamente," while in the journal's table of contents [4; Vol. I] on page 338 — "nuovamente." — *The footnote is ours = Footnote ours.*



Page from *Constitutum usus pisanae civitatis*
Footnote at the bottom (referring to the 26th line) is a decree about Fibonacci's annual remuneration or salary

Pisan document text

> "Considerantes nostre civitatis et civium honorem atque profectum, qui eis, tam per doctrinam quam per sedula obsequia discreti et sapientis viri magistri Leonardi Bigolli, in abbacandis estimationibus et rationibus civitatis eiusque officialium et aliis quoties expedit, conferuntur; ut eidem Leonardo, merito dilectionis et gratie, atque scientie sue prerogativa, in recompensatione laboris sui quem substinet in audiendis et consolidandis estimationibus et rationibus supradictis, a Comuni et camerariis publicis, de Comuni et pro Comuni, mercede sive salario suo, annis singulis, libre xx denariorum et amisceria consueta dari debeant (ipseque pisano Comuni et eius officialibus in abbacatione de cetero more solito servat), presenti constitutione firmamus."
> [3; p. XIII]

Professor David Breyer Singmaster translates this decree as follows [49; 2) Bibliographical Material]:

> "Considering the honour and progress of our city and its citizens that is brought to them through both the knowledge and the diligent application of the discreet and wise Maestro Leonardo Bigallo in the art of calculation for valuations and accounts for the city and its officials and others, as often as necessary; we declare by this present decree that there shall be given to the same Leonardo, from the Comune and on behalf of the Comune, by reason of affection and gratitude, and for his excellence in science, in recompense for the labour which he has done in auditing[4], and consolidating[5] the above mentioned valuations and accounts for the Comune and the public bodies, as his wages or salary, 20 pounds in money each year and his usual fees (the same Pisano shall

[4] Audit — independent evaluation of a person, organization, system or process. Financial audit – review of the financial statements of a company resulting in the publication of an independent opinion on the accuracy and completeness of the statements. — *Footnote ours.*

[5] Consolidation – the combination of several actions into one; the merger or joining of several financial documents or firms into single one. — *Footnote ours.*

> continue to render his usual services to the Comune and its officials in the art of calculation etc.)”[6]

A translation of the Pisan decree into German:

> In Anbetracht unserer Stadt und der Bürger Ehre und Vorteil, der ihnen wie oft schon bei Bedarf zustatten kommt sowohl durch die Gelehrsamkeit als auch durch die emsigen Dienste des ausgezeichneten und klugen Mannes und Lehrers Leonardo Bigollo, die im Berechnen von (Steuer-)Schätzungen und Rechnungen für die Stadt und ihre Amtsträger und anderem bestehen, setzen wir durch vorliegende Konstitution fest, dass eben diesem Leonardo aus Wertschätzung und Gunst, aufgrund des Verdienstes und aufgrund des Vorrangs seiner Kenntnis zum Ausgleich für seine Arbeit, die er ausführt durch Prüfung und Feststellung oben genannter Schätzungen und Rechnungen, von der Gemeinde und ihren Kämmerern — von der Gemeinde berufen und für die Gemeinde handelnd — als Lohn bzw. sein Gehalt jährlich XX Pfund Pfennige und die üblichen Naturralleistungen gegeben werden müssen und dass er der Gemeinde von Pisa und ihren Amtsträgern fortan wie gewohnt durch Ausführung von Rechnungen dient.

This German translation is taken from the book “Von Zahlen Und Größen” by Lüneburg, 2008, Band 1, p. 318.

In 1865 (see Fig. 1) a large commemorative marble plaque bearing this inscription in Latin with an appropriate heading was placed in the atrium of the *Archivio di Stato* in Pisa.

The facsimile of this plaque is provided by the medieval historian of mathematics professor Gino Arrighi (1906–2001) in the introduction [1; Introduzione, p. 15] to his Italian translation (under the title of “Leonardo Fibonacci. La Pratica di Geometria”) of Fibonacci’s work “Practica Geometriae” (*The Practice of Geometry*).

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[6] In our opinion “20 pounds” in Singmaster’s translation should be “20 *denarii*” or “20 *denariuses*,” which is closer to the original Latin text, and not “pounds.” Same goes for Lueneburg’s “*XX Pfund Pfennige*.” — *Footnote ours*.

Autobiography

At the very beginning of *Liber Abaci* — in the publication of Boncompagni of 1857 [7; vol. I, p. 1] and in translation by Laurence Sigler (1928–1997) [44; p. 15] this is the second paragraph from the beginning — Leonardo gives us some information about his earlier life.

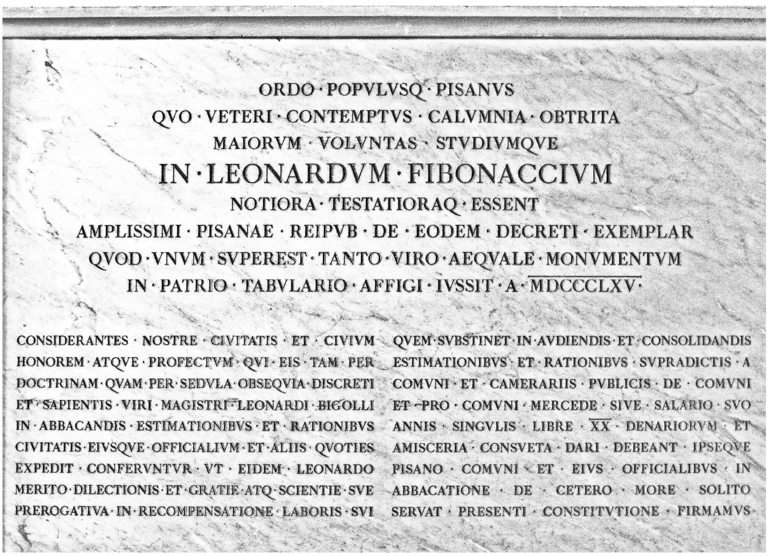

Fig. 1. Wall plaque in the entrance hall of the *Archivio di Stato di Pisa* (marble frame is shown only on the top)

The second paragraph in interpretation by professor Richard E. Grimm (1926–1999), who comments on it in detail in his article "The autobiography of Leonardo Pisano" and provides the translation of it from Latin to English, is provided below [19].

The second paragraph in Grimm's interpretation

Cum genitor meus a patria publicus scriba in duana bugee pro pisanis mercatoribus ad eam confluentibus constitutus preesset, me in pueritia mea ad se venire faciens, inspecta utilitate et commoditate futura, ibi me studio abbaci per aliquot dies stare voluit et doceri. Vbi ex mirabili magisterio in arte per novem figuras indorum introductus, scientia artis in tantum mihi pre ceteris placuit, et intellexi ad illam quod quicquid studebatur ex ea apud egyptum, syriam, greciam, siciliam, et provinciam[7] cum suis variis modis, ad que loca negotiationis causa postea peragravi per multum studium et disputationis didici conflictum. Sed hoc totum etiam, et algorismum atque artem pictagore quasi errorem computavi respectu modi indorum. Quare, amplectens strictius ipsum modum indorum et attentius studens in eo, ex proprio sensu quedam addens et quedam etiam ex subtilitatibus euclidis geometrice artis apponens, summam huius libri, quam intelligibilius potui, in quindecim capitulis distinctam componere laboravi, fere omnia que inserui certa probatione ostendens, ut extra perfecto pre ceteris modo hanc scientiam appetentes instruantur, et gens latina de cetero, sicut hactenus, absque illa minime inveniatur. Si quid forte minus aut plus iusto vel necessario intermisi, mihi deprecor indulgeatur, cum nemo sit qui vitio careat et in omnibus undique sit circumspectus. [19; p. 100]

Grimm's translation

After my father's appointment by his homeland as state official in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and, in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days. There, following my introduction, as a consequence of mar-

[7] Province (Lat. *provincia*) — 1) a territory outside Italy, conquered and annexed by the Roman Empire and governed by a legate or procurator as a unit of the empire; later usually Asia Minor, Africa; 2) south-east Gaul (Lat. *Gallia*); later region Provence in the south-east of France. — *Footnote ours.*

> velous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business, I pursued my study in depth and learned the give-and-take of disputation. But all this even, and the algorism, as well as the art of Pythagoras I considered as almost a mistake in respect to the method of the Hindus. Therefore, embracing more stringently that method of the Hindus, and taking stricter pains in its study, while adding certain things from my own understanding and inserting also certain things from the niceties of Euclid's geometric art, I have striven to compose this book in its entirety as understandably as I could, dividing it into fifteen chapters. Almost everything which I have introduced I have displayed with exact proof, in order that those further seeking this knowledge, with its pre-eminent method, might be instructed, and further, in order that the Latin people might not be discovered to be without it, as they have been up to now. If I have perchance omitted anything more or less proper or necessary, I beg indulgence, since there is no one who is blameless and utterly provident in all things. [19; p. 100]

Grimm's Latin version is based on a thorough analysis of the six known copies of the manuscript *Liber Abaci* [19; p. 99, p. 101], [49; 2) Biographical Material].

scriba

Publicus scriba is how Leonardo indicated the occupation of his father in *Liber Abaci*.

The French historian and an expert on the relations between Europe and the East in the Middle Ages Jacques Marie Joseph Louis, Count of Mas Latrie (1815–1897) writes [37; Introduction historique, pp. 130–131] that the Republic of Pisa had two permanent consuls on the African continent, one residing in Tunis, and the other — in Bejaia (Bougie). Independently from the consul and his employees, and other legal officials, each nation had a special official, or a Christian book keeper, called the *scribe* ("teneur de livres chrétien, appelé l'*écrivain*"). He was charged to collect customs duties, register merchants' accounts of his country to present to the Muslim Customs ("*duane arabe*"), and also watch for his compatriot interests.

Mas Latrie notes [37; Introduction, p. 131]:

> "C'est auprès de son père, écrivain de la nation pisane à la douane de Bougie, à la fin du douzième siècle[4], que le célèbre mathématicien Léonard Bonacci de Pise, plus connu sous le nom de Fibonacci, apprit les principes de l'arithmétique, de l'algèbre et de la géométrie.
>
> ___
>
> [4] " Publicus scriba. " "
>
> (It is with his father, the public scribe[4] of the Pisan nation at the customs of Bejaia, at the end of the 12th century, that the famed mathematician Leonardo Bonacci of Pisa, better known under the name of Fibonacci, learnt the principles of arithmetic, algebra and geometry.
>
> ___
>
> [4] " Publicus scriba. ")

Some authors interpret *publicus scriba* as notary or clerk (*notaire*; *notaio*; *Notarius*; *Cancelliere*) — see Boncompagni [5; pp. 6–7].

Grimm states [19; p. 101]:

> "We simply do not know the precise nature of the position held by Leonardo's father. He was appointed (constitutus) by Pisa to this post, which certainly involved duties at Bugia (present-day Bugie in Algeria) in connection with the Pisan duana, a word which we perhaps translate too easily as customshouse. The text as it stands offers no basis for much of the standard lore found in biographies of Leonardo regarding his father as "secretary," "merchant," "agent," "business man," "head of factory," "warehouse head," etc."

Translated from Latin *scriba* (pronounced 'scri-ba) means a scribe, penman, clerk, secretary. Scriba is an ancient profession; it was a person who could read and write and usually performed secretarial or administrative duties. Scribes in ancient Rome represented the highest class of the magistrates' attendants and thus were different from so-called *librarii* (copyists). Sometimes scribes were chosen from among senators. The position of *scribes* in the province was considered an honourable one[8].

Professor Alfred E. Lieber states the following [29; p. 239]:

> "… the scribes obtained a powerful hold over the economy, which they were subsequently reluctant to surrender. At the end of the thirteenth century Pisa counted 232 scribes, Genoa 200, and Florence 600, …"

duana

Duana [7; vol. I, p. 1] or *duhana* [6; p. 3] is another word from the autobiography of Leonardo with an unclear meaning. Usually the combination of words *duana bugee* is translated as a c*ustoms-house in Bejaia* [19; p. 101]. In such government institutions officials processed the paperwork for the import and export of goods and collected customs duties.

Lueneburg provides the following versions of the word *duana*: *dogana*, *al-diwan*, *diwano*, *Diwan* [35; pp. 21−22].

In Dahl's[9] dictionary there is a similar sounding word "дуван" [du'van], one of the meanings of which is a customs house on the Caspian seashore.

According to Alfred Lieber [29; p. 237]:

> "Each foreign merchant arriving in a Muslim port had to clear his goods through the Customs, which was known as the *dīwān* in Arabic, while its storehouse was the *makhzen*. (These two terms later came into everyday use in medieval European commerce as, respectively, *dohana*, *dogana*, or *douane* and *magasin* or ″magazine″.)"

8 Mommsen, Theodor. *Römisches Staatsrecht*. – Leipzig, 1876. – Bd. I, S. 331.
9 Dahl, Vladimir Ivanovich. *Explanatory Dictionary of the Live Great Russian Language* (Tolkouvij slowar' …). Moscow : "Citadel," 1998.

Period of study

> “How much time he actually spent at Bugia in his study Leonardo does not tell us. ⟨…⟩ Just who gave Leonardo this "marvelous instruction" is not stated. It has been frequently assumed that his instructor was Moorish, but there is no hint of this in the text.” [19; p. 101]

bugee

Bejaia (*bugee* in Leonardo Pisano’s writings), Béjaïa (Bédjaïa) in French or Bgayet in Kabyle (spoken language in some Provinces of Algeria) is a small and very ancient city on the northern Algerian coast of the Mediterranean Sea. It was formerly known under various European names, such as Budschaja in German, Bugia or Bougia in Italian, Bougie [bu'ʒi] in French or Bidjaya.

The city was an important port and cultural center in the time of Fibonacci — learning here was honored as were the pursuit of industry and commerce. It traded with many cities, with western and eastern Africa and with Sahara. Ships unloaded here, caravans came here by land, and it was a big Mediterranean trading hub. Merchants from everywhere could be found in the city. Bejaia had excellent trade partnerships with the Italian cities, especially with Pisa, one of the most economically developed cities of the Middle Ages.

Names in the times of Fibonacci

There were no last names (family names) in the times of Leonardo. People were referred to by name and the place of their birth. Thus during his lifetime Leonardo was called Leonardo Pisano or Leonardo da Pisa. In Italian Leonardo Pisano means *Leonardo the Pisan*, while Leonardo da Pisa means *Leonardo of Pisa*. In Latin Leonardo Pisano is *Leonardus Pisanus*.

To more easily distinguish between them, people were given nicknames, which reflected their character traits, or people assigned to themselves some, often, prominent lineage. Such references to illustrious ancestry were then a common practice in Italy.

About Leonardo's names

On how Leonardo called himself, or how the scribes of his manuscripts called him, we can judge by the titles of works, containing his name, or by how his name was written in the texts.

For example:

• ***Leonardus filius Bonaccij Pisani*** *Michaeli Scotto summo philosopho* [6; p. 3] (Leonardo, son of Bonacci the Pisan, to Michael Scott, the greatest philosopher);

• *Incipit liber Abaci Compositus a* ***leonardo filio Bonacij Pisano*** *In Anno. M cc ij* [7; vol. I, p. 1], which can be translated as "Here begins *The Book of the Abacus* composed in the Year 1202 by Leonardo, son of Bonacci the Pisan," if the word "*Abaci*" can be formally translated as "*Abacus*."

• *Incipit pratica geometriae composita a* ***Leonardo pisano de filijs bonaccij*** *anno .M.° cc.° xx.°* [7; vol. II, p. 1] (Here begins *The Practice of Geometry*, written in 1220 by Leonardo the Pisan of the sons of Bonacci);

• *Incipit Abbacus* ***Leonardi de domo filiorum bonacii pisani*** *compositus A. M. CC. II. et correptus*[10] *ab eodem A. M. CC. XXVIII* [5; p. 25]. (Here begins *Abbacus* by Leonardo of the sons of the family (kin) of Bonacci the Pisan, written in 1202 and corrected by him in 1228). The word *Abbacus* is translated here by us formally as *Abbacus*.

Bonaini provides the following variants of the names [3; p. XV], [4; p. 242]: ***Leonardus filius Bonaccii***, ***Leonardus ex filiis***, ***Leonardus Bigollosius filius Bonaccii***.

From the examples presented here it follows that either Leonardo is the son of Bonacci and therefore Leonardo's father was called Bonacci, or Leonardo is one of the sons of Bonacci and therefore Leonardo had brothers, or Leonardo comes from the family of Bonacci in which case the father's name is not Bonacci.

[10] The work is known by the titles that use *correptus* and *correctus* (Lat. – to correct, to improve); for example, *correctus* is provided on pages 31 and 45 of Boncompagni's "*Della vita ...*" [5]. — *Footnote ours*.

This situation — with an interpretation by historians of variants of Leonardo's names — Boncompagni details in his work "Della vita …" [5; pp. 7−21].

bigolli, bigollus, Bigollo

In the same work [5; pp. 16–22] Boncompagni provides the word "Bigollo," which occurs as part of Fibonacci's name, and various forms of *Bigollo* such as *Bigollone*, *Bigolloso*, *Bighellone*. For example:

- *Incipit pratica geometrie composita a* ***leonardo Bigollosie filio Bonacij Pisano*** *in Anno M. CC. XXI.* [5; p. 16];

- *Incipit practica Geometriae composita a* ***Leonardo Pisano Bigollo*** *Filiorum* [5; p. 19].

In Fibonacci's work "Flos" his name is presented as follows:

- *Incipit flos* ***Leonardi bigolli pisani*** *super solutionibus quarumdam questionum ad numerum et ad geometriam, uel ad utrumque pertinentium* [7; vol. II, p. 227].

The same name — ***Leonardi bigolli pisani*** — we can see in provided by us earlier *Pisan document*. In both cases *Leonardi bigolli pisani* is used in the genitive case (*genetivus*) in Latin; in nominative case (*nominativus*) it would be *Leonardus bigollus pisanus*. Italians in nominative case write this name as *Leonardo Bigollo Pisano*.

The differences in the spelling of *Bonacij*, *Bonaccii* or *Bonacci* can be partly explained by the mixture of spoken Italian and written Latin, which was a common practice in the times of Leonardo.

In Italian *Bonacci* is also the plural of *Bonaccio*.

Many unsuccessful attempts have been made to explain the meaning of *Bigollo* [45; p. xv]. To this day it is not clear what Leonardo meant by calling himself in that way.

For example, professor David Eugene Smith (1860−1944) and professor Louis Charles Karpinski (1878−1956) in their book “The Hindu-Arabic Numerals,” referring to the reprint (*estratto*) of the article by Gaetano Milanesi (1813−1895) “Documento inedito …” [40], provide the following version [50; p. 130]:

> “Milanesi … has shown that the word *Bigollo* (or *Pigollo*) was used in Tuscany to mean a traveler, and was naturally assumed by one who had studied, as Leonardo had, in foreign lands.”

For Milanesi’s version see page 83 in *Documento inedito* … [40].

One can find speculations that *Bigollo* means a *good-for-nothing*, *absent-minded* person. To which Bonaini reasonably remarks that it is unlikely that the word *Bigolli*, added to the name of Leonardo, could mean a pejorative nickname, if it was in the very same solemn decree, which was passed to honor such a distinguished countryman:

> “… la parola *Bigolli*, unita al nome di Leonardo, non può credersi denotare un appellativo di dispregio, trovandosi essa in quello stesso solenne decreto che era inteso ad onorare questo insigne concittadino.”[5; p. XV]

Fibonacci

In the 19th century there were attempts to figure out what the word *Fibonacci* meant, and in the 20th century — where it came from.

Laurence Sigler, for example, writes [45; pp. xv-xvi]:

> “It is, however, quite worth saying that the use of the sobriquet *Fibonacci* for Leonardo Pisano probably originated with the mathematical historian Guillaume Libri[11] in 1838; there is no evidence that Leonardo so referred to himself or was ever so called by his contemporaries … Nevertheless,

[11] Guglielmo Libri, Count (1803−1869; in Fr.: *Guillaume*) or Guglielmo Icilio Timoleone Libri, Conte Carrucci della Somaia, or Guglielmo Bruto Icilio Timoleone Libri Carucci dalla Sommaia (conte) — an Italian aristocrat, mathematician and bibliographer, professor. Numerous and authoritative works won him respect of the whole scientific world. — *Footnote ours*.

> as has been often the case with mathematical history, these lamentable errors or fantasies catch on, persist, and seem never to be correctable."

Others, as, for example, the known historian of mathematics professor Victor J. Katz, suppose that nickname *Fibonacci* was made up by Baldassarre Boncompagni [23; p. 299]:

> "Leonardo, often known today by the name Fibonacci (son of Bonaccio) given to him by Baldassarre Boncompagni, ..."

However Boncompagni in his work *Della vita* lists the authors who use the word "Fibonacci" in their works, as well as the authors who not only use it, but also explain its origin [5; pp. 8–10]. These authors are provided below (pages from *Della vita* we put in parentheses; publication dates of authors, mentioned by Boncompagni, we put in square brackets; for more details about these mentioned publications see *Della vita* [5]).

- Authors who mention the name *Fibonacci* in their works:

(p. 8): Giovanni Gabriello Grimaldi [1790 (t. I; pp. 161–219)];
(p. 8): Guillaume Libri [1838, 1840, 1841 (t. II, t. III, t. IV)];
(p. 8): Michel Chasles [1837];
(p. 9): Nicollet [1815 (t. XIV, p. 481)];
(p. 9): Gartz [1846 (Section 1, Theil 43, pp. 444 – 446)];
(p. 9): Augusto de Morgan (Augustus de Morgan) [1847];
(p. 10): Ranieri Tempesti [1787].

- Authors who provide the explanation of the name *Fibonacci* in their works:

(p. 8): John Leslie [1820; p. 227]: Fibonacci = son of Bonacci (figliuolo di Bonacci);
(p. 8): Pietro Cossali [1799 (vol. II)]: Fibonacci = filius + Bonacci;
(p. 9): Flaminio Dal Borgo [1765]: Fibonacci = filio + Bonacci;
(p. 9): Girolamo Tiraboschi [1806; p. 170], [1823; p. 254]: Fibonacci = figliuol + Bonaccio;

(p. 10): Giovanni Andres [1812]: Fibonacci = figlio + Bonaccio;
(p. 10): Giovanni Gabriello Grimaldi [1790 (t. I; p. 163)]: Fibonacci = filio + Bonacci;
(p. 10): Guillaume Libri [1838 (t. 2, p. 20)]: Fibonacci = filius + Bonacci.

As we can see Flaminio Dal Borgo (1705–1768) explained the origin of the family name *Fibonacci* in 1765 — long before Boncompagni and Libri.

Since the family name or nickname *Fibonacci* was known to a wide enough circle of authors, one can suppose that they based their claims on earlier sources known to them. Bonaini, for example, provides the following note of the Roman Empire notary Perizolo, in which Perizolo already in 1506 mentions Leonardo as "Fibonacci" [3; pp. XI–XII] (see also: [42; p. 388]):

> "Lionardo Fibonacci fue nostro concive, e vivette nelli anni 1203. Vidde tutto el mondo; tornoe a Pisa, e recò i numeri arabichi e l'aritmetica, e ne compose un libro che in questo tempo, dello anno 1506 pisano, nello tempo scrivo, tiene la famiglia delli Gualandi, e vi sono expressi li numeri fino al decimo, quale composto forma la decina, et insegna contare el …"
>
> (Leonardo Fibonacci was our countryman and lived in the year 1203. He saw all the world, returned to Pisa and brought with him Arabic numerals and arithmetic, and from this composed a book, which in 1506 by the Pisan style calendar, at the time of this writing, can be found in the family of Gualandi. The numbers up to ten in the book compose the ten (group of ten) and it teaches counting …)

Boncompagni in his work "Della vita …" [5; pp. 84–85] also provides this record with reference to Bonaini.

The interpretation of the name *Fibonacci* (fi + Bonacci) as *son of Bonacci*, formed from the shortened Italian *figlio* (son) or Latin *filius* (son) and *Bonacci*, became widespread, however Boncompagni was against such an interpretation. The prefix *Fi* occurs quite often in the Tuscan names — such as *Figiovanni*, *Fighineldi*, *Firidolfi*, *Fifanti* and similar others. Boncompagni constantly pointed out that in these kinds of names *Fi* comes from the word *filiis* and concluded by analogy, that *Fibonacci* also comes from *filiis* and *bonacci* [5; pp. 10–15]. Thus *Fibonacci* does not mean *son of Bonacci* but *from the family of Bonacci*. For example, the name "Johnson" (John-son) means "A Son of The John-family," but not "A son of John."

Leonardo's father

What was *Bonacci* — Leonardo's family name or the name of his father? Disputes regarding this took place until a well-known Italian scholar Gaetano Milanesi, who wrote much about history of Italian art, discovered a notarial act of purchase and sale from 28th of August 1226 [40]. The document begins [40; p. 87]:

> " 1226, 28 *d'Agosto* (*).
>
> In nomine domini Amen. Dominice Incarnationis Anno Millesimo ducentesimo vigesimo sexto, Indictione tertia decima, quinto Kalendas Septembris. Ex hoc publico instrumento omnibus sit manifestum, quod Bartholomeus quondam Alberti Bonacii vendidit et tradidit *Leonardo bigollo quondam Guilielmi*, procuratori et certo nuntio Bonaccinghi germani sui quondam suprascripti Guilielmi; ut apparet in sceda procurationis rogata a Pagano notario quondam Malagallie, et a me Bonafidanza notario visa et lecta; …
>
> ———————————
>
> (*) Archivio Centrale di Stato di Firenze — Sezione del Diplomatico. Carte degli Olivetani di Pisa. "

The main point from the above excerpt is: Bartholomeus is selling, and Leonardo and his brother Bonaccingus, both descendants of one and the same *Guilielmi*, are buying (see also: [49; 2) Biographical Material] and [50; p. 129]).

In the notarial act composed in Latin, the name *Leonardo* can be found in the following forms: *eidem Leonardo*, *suprascripto Leonardo*, *suprascripti Leonardi*, *suprascriptum Leonardum*, *quondam Leonardi* and in the already presented above form

Leonardo bigollo. The names *Leonardo bigollo*, *Guilielmi* and *Bonaccinghi* from the quoted excerpt are written in the nominal case in Latin as *Leonardus bigollus*, *Guilielmus* and *Bonaccingus* respectively. The spelling of the name *Bonaccinghi* in the form *Bonaccingus* is encountered later in the text of this notarial act: "*… nomine pro predicto Bonaccingo germano suo, et ipse Bonaccingus et heredes dicti Bonaccinghi eorum directo et utili nomine, …*" [40; p. 87].

Italians and everyone after them spell all the names, in one way or another connected to Fibonacci, in the Italian manner and that is why the name of Leonardo's father is spelled as *Guglielmo* and not *Guilielmus*, as in Latin; in English it is sometimes written *William* [50; p. 129]. The name *Guglielmo* is used by Milanesi in his publication of 1867 — before setting forth the contents of document discovered by him— the notarial act of purchase and sale [40; p. 82]:

> "Ma ogni disputa ed incertezza intorno a questo è tolta ora di mezzo dal presente documento, il quale ci scopre che Lionardo nacque da un Guglielmo, ed ebbe un fratello per nome Bonaccingo."
>
> (But any disputes and uncertainties around this question are dismissed by the present document which revealed to us that Lionardo was born of Guglielmo and had a brother by the name of Bonaccingo.)

Chapter II

FIBONACCI'S WORKS

It is known that Fibonacci wrote the following:

- **Liber Abaci** (1202; 1228) — "The Book of Calculation" [7; vol. I, pp. 1−459]. *Liber abaci* is an encyclopedic work, both theoretical and practical, treating much of the known mathematics of the thirteenth century on arithmetic, algebra and problem solving [44; p. 4]. The revised version of 1228 is dedicated to Michael Scot [6; p. 89], [7; vol. I, p. 1], a scholar at Frederick's court.

- **Practica Geometriae** (1220 or 1221) — "The Practice of Geometry" or "Practical Geometry," or "Applied Geometry" [7; vol. II, pp. 1−224]. The other great work of Fibonacci. This contains a large variety of theorems with proofs, geometric and trigonometric problems. The work was apparently based on the lost book of Euclid *περί διαιρέσεων* (*On Divisions of Figures*), and also on Heron's *Metrica*. [43; p. 612]. *Practica Geometriae* is dedicated to another court scholar Domenico [6; p. 96], [7; vol. II, p. 1].

- **Liber Quadratorum** (1225) — "The Book of Squares" or "The Book of Square Numbers" [7; vol. II, pp. 253−283]. This is a book on advanced algebra and number theory, and probably Fibonacci's most impressive work. It is dedicated to the Emperor Frederick II [6; pp. 25–26, p. 87].

- **Flos** (around 1225) — "The Flower." In this short work [7; vol. II; pp. 227–247], the title of which might suggest that algebra (figuratively) is the "flower" or "blossom" of Mathematics, Fibonacci provides solutions to a number of problems.

- **Di minor guisa** or *Liber minoris guise* is a book on commercial arithmetic [22; p. 25], [6; p. VI, p. 242, p. 248]. Lost. In his *Liber Abaci* Fibonacci mentions it thus [7; vol. I, p. 154]: "…quem in libro minoris guise docuimus, …"

- ***Commentary** on Book X of Euclid's "Elements"* (*Elementa* in Latin) or *The Comment to the Tenth Book of Euclid's Elements* [6; pp. 246−248]. Lost. The *Commentary* contains a numerical treatment of irrational numbers.

- **Epistola ad Magistrum Theodorum** (around 1225) — A letter to Master Theodorus, philosopher at the court of Frederick II [7; vol. II; pp. 247−252]. It contains solutions to a number of problems.

Fibonacci lived in the days before printing, so his treatises were hand written and the only way to have a copy of his manuscript was to have another hand-written copy made.

Most of our present-day knowledge on Fibonacci comes from the extensive research of Baldassarre Boncompagni. He reviewed manuscripts in many libraries and brought together his findings in a series of works in the 1850s. Besides Boncompagni (1821–1894) it is necessary to note in this regard Pietro Cossali (1748–1815), Giovanni Battista Guglielmini (or Giambattista Guglielmini; 1763 – 1817), Guglielmo Libri (1803 – 1869) and Francesco Bonaini (1806 – 1874).

Cossali, for example, in the first volume of his 1797 work "Origine, trasporto in Italia, primi progressi in essa dell'algebra" (*Origin, Transmission to Italy, and Early Progress of Algebra There*) devotes a lot of space to Fibonacci and believes that he played an important role in borrowing from the Arabs, delivering to Italy and spreading algebra in Italy and Europe. Cossali was familiar with Fibonacci's works *Liber Abaci* and *Liber Quadratorum* and in the first volume of "Origine …" he analyses some problems from these works.

Regular mentions of Fibonacci, as we have shown above with reference to Boncompagni [5; pp. 8–10], began to appear in the end of the 18th and beginning of the 19th century.

Frederick II and problems for Fibonacci

From Fibonacci's works it can be seen that he was known at the court of Emperor Frederick II and had contacts with such court scholars as the Master philosopher John of Palermo[12] (magister Johannes panormitanus philosophus — in Latin; maestro Giovanni Palermitano or Giovanni da Palermo, filosofo — in Italian [6; p. 7, p. 29, p. 106]); Michael Scot or Scott[13] (magister Michael Scotus, philosophus et astrologus; maestro Michele Scoto, filosofo ed astrologo — in Latin and Italian respectively [6; p. 89]); Master Theodore of Antioch, philosopher and astrologer (magister Theodorus Antiochenus, philosophus et astrologus; maestro Teodoro di Antiochia filosofo ed astrologo — in Latin and Italian respectively [6; p. v, p. 77, p. 78]) and Master Domenico from Spain (magister Dominicus Hispanus; maestro Domenico — in Latin and Italian respectively [6; p. v, p. 98, p. 105], [7; vol. II, p. 253]).

In approximately 1225, when Frederick II held his court at Pisa, Fibonacci was granted an audience. The audience was obtained through Domenico, as Fibonacci himself points out in *Liber Quadratorum* [8; p. 55] & see [6; p. 25], [7; Vol. II, p. 253], [45; p. 3]:

> "Cum Magister Dominicus pedibus celsitudinis vestre, princeps gloriosissime domine F., me Pisis duceret presentandum, …"

---

[12] Iohannes panormitanus [7; vol. II, p. 227] or Johannes panormitanus [8; p. 2] is John the Palermitan or John of Palermo in English. Panormus, Panormum today is the Palermo city. In Boncompagni's texts, written in Italian, Johannes panormitanus is called Giovanni Palermitano [6; p. 29] or Giovanni da Palermo [6; p. 106]. He is mentioned also as a notary (*notarius*) in several diplomatic documents of Frederick II.

[13] Scottish scholar Michael Scot (1175−1234) was a philosopher, astrologer and translator at the court of Emperor Frederick II. He is called Michael Scotus in his manuscript *Liber Phisionomiæ*, which begins with the words: "*Incipit Liber Phisionomiæ: quem compilauit magister Michael Scotus ad preces. D. Federici romanorum imperatoris*."

> (When in Pisa Master Domenico brought me to the feet of Your Highness, the Most Illustrious Lord F., (Frederick II — *remark is ours*) and introduced me, ...)

Leonardo calls Domenico his friend in the dedication at the beginning of his manuscript *Practica Geometriae*:

> "Rogasti me Amice Dominice et reuerende magister: ut tibi librum in pratica geometriae conscriberem." [6; p. 96]; see also [7; Vol. II, p. 1]
>
> (By your request, my friend Domenico and honorable Master, the book *The Practice of Geometry* I dedicate to you.)

During the audience with the emperor, John of Palermo presented Fibonacci with a series of advanced mathematical problems — this resulted in a sort of a mathematical tournament or trial [53; p. 110].

Fibonacci refers to these events in the beginning of his work *Flos* and provides three problems [6; p. 4] from the ones presented to him by John of Palermo. For one of these problems Fibonacci provides the solution and answer, for the two others — only the answers, however the solution for one of these two problems is presented in *Liber Quadratorum*.

Flos

The full title of this short manuscript is: "Flos Leonardi bigolli pisani super solutionibus quarumdam questionum ad numerum et ad geometriam, uel ad utrumque pertinentium" [7; vol. II; pp. 227−247] (see also [8; pp. 1−43]). Title of the manuscript can be interpreted in the following sense: "*Flower* by Leonardo Bigollo Pisano or blossom of the best solutions of certain questions pertaining to arithmetic or geometry or both."

Leonardo dedicated his *Flos* to Frederick II [6; p. 20]. When Cardinal Raniero Capocci (Cardinale Raniero Capocci di Viterbo; 1180 or 1190−1250) learned of Leonardo's work and requested a copy from the author, Leonardo somewhat modified the content, adding a few other problems of a similar type and a dedication to the cardinal [6; pp. 17−19, p. 94].

Following the prologue (*Explicit prologus incipit tractatus eiusdem*) the manuscript begins thus [8; p. 2]; [7; vol. II, p. 227]):

> Cum coram maiestate vestra, gloriosissime princeps Frederice, magister Johannes panormitanus, phylosophus vester, pisis mecum multa de numeris contulisset, interque duas questiones, que non minus ad geometriam quam ad numerum pertinent, proposuit, quarum prima fuit ut inveniretur quadratus numerus aliquis, cui addito vel diminuto quinario numero, egrediatur quadratus numerus, quem quadratum numerum, ut eidem magistro Johanni retuli, inveni esse hunc numerum, undecim et duas tertias et centesimam quadragesimam quartam unius. Cuius numeri radix est ternarius et quarta et VI.[a] unius, cui quadrato numero si addantur quinque provenient XVI, et due tertie et una centesima quadragesima quarta, qui numerus est quadratus, cuius radix est quatuor et una duodecima. Item si auferantur V ab eodem quadrato numero, remanebunt VI et due tertie et una centesima quadragesima quarta, qui numerus etiam quadratus est, cuius radix est duo et tertia et quarta unius.
>
> (In Pisa, in the presence of Your Majesty, the Most Glorious Lord Frederick, magister John of Palermo, a philosopher of Yours, posed to me a series of mathematical problems among which two pertain no less to geometry than to arithmetic (*numerum*). The first of them (*quarum prima fuit* …) is to find a square number which when increased or diminished by five gives a square number. The square of this number, about which John, who thought of it, is asking, is eleven and two thirds and one one hundred and forty-fourth (*undecim et duas tertias et centesimam quadragesimam quartam unius*), the root of which is three and a quarter and one sixth (*ternarius et quarta et VI.[a] unius*), which when squared and after adding five is sixteen and two thirds and one one hundred and forty-forth, this number being a square has a root of four and one twelfth. Similarly if we subtract 5 from the same square of a number, we get 6 and two thirds and one one hundred and forty-fourth, which number is also a square, the square root of which is two and a third and a fourth.)

By using the modern symbolic notation system the problem formulation and the solution can be written as follows:

$$\begin{cases} x^2+5=y^2 \\ x^2-5=z^2 \end{cases}$$

$$\begin{array}{ll} x^2 = 11+2/3+1/144, & x=3+1/4+1/6; \\ y^2 = x^2+5=16+2/3+1/144, & \mathrm{y}=4+1/12; \\ z^2 = x^2-5=6+2/3+1/144, & z=2+1/3+1/4. \end{array}$$

As we can see Fibonacci does not provide the solution to the problem, however the reason for this becomes clear from the rest of the text. Fibonacci says that, he was thinking a long time about the solution to this question and noticed a certain general properties of squares of numbers. The material, gathered as a result, affords an opportunity to write a small work on the properties of square numbers under the title "Little Book of Squares" (*Libellum Quadratorum*) [53; p. 113] (see also [7; vol. II, p. 227], [8; p. 3]):

> "Et cum diutius cogitassem unde oriebatur predictae questionis solutio, inveni ipsam habere originem ex multis accidentibus, quae accidunt quadratis numeris, et inter quadratos numeros; quare hinc sumens materiam, libellum incepi componere ad Vestre Majestatis Celsitudinis gloriam, quem Libellum Quadratorum intitulavi."

Ultimately Fibonacci's work was titled "Liber Quadratorum" and it contained, in particular, the examination of the systems of equations of type $x^2+a=y^2$; $x^2-a=z^2$:

> "Invenire numerum, quo addito super quadratum numerum, et diminuto ab ipso, faciat semper quadratum numerum, …" [8; p. 83].
>
> (Find a number, that when added to a square or subtracted from it always results in a square.) [14]

[14] More detailed analysis of this system of equations from *Liber Quadratorum* can be found in Boncompagni's 1855 work "*Intorno alla risoluzione delle equazioni simultanee* $x^2+h=y^2$, $x^2-h=z^2$."

Here [8; pp. 96−98] Fibonacci also provides the solution to the problem proposed to him during the audience with the emperor:

> "Volo invenire quadratum, cui addito 5 vel diminuto, faciat quadratum numerum."
>
> (Find a square that when increased or decreased by five results in a square.)

So we are indebted to the audience with the emperor for providing us with *Liber Quadratorum*, as well as *Flos* and *Epistola* according to professor Olry Terquem (1782−1862), who notes this in his work "Sur Léonard Bonacci de Pise et sur trois écrits de cet auteur publiés par Balthasar Boncompagni" (*About Leonardo Bonacci of Pisa and three works of this author published by Balthasar Boncompagni*) [53; p. 110].

The second of the problems:

> "Altera vero questio a praedicto magistro Johanne proposita fuit, ut inveniretur quidam cubus numerus, qui cum suis duobus quadratis et decem radicibus in unum collectis essent viginti …" [6; p. 6]; [8; p. 3].
>
> (The other question, by the aforementioned Master John, was to find a number cubed, which together with the doubled square of this number and ten of its roots, gives twenty.)

Thus it was required to solve the cubic equation:

$$x^3 + 2x^2 + 10x = 20.$$

Fibonacci does not provide the solution for this problem, however the most notable in this case is the thorough analysis conducted by him[15], where he shows that the problem does not have a solution for whole and rational numbers, and that the root cannot be one of the quadratic irrationalities discussed in Euclid's *Book X* of his *Elements*. Such an analysis was a unique occurrence for the

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[15] John Derbyshire notes this (pp. 69−70) in his book *Unknown Quantity: A Real and Imaginary History of Algebra*. — Washington, D.C. : Joseph Henry Press, 2006. — 374 p. + [8 p.]

medieval algebra. Further Fibonacci indicates, that the problem can be solved with high accuracy by an approximation and provides an answer without explaining the method behind it [6; p. 6] (see also [7; Vol. II, p. 234]):

> "… esse unum et minuta XXII., et secunda VII., et tertia XLII., et quarta XXXIII.[16], et quinta IIII , et sexta XL."

This is:

$$x = 1 + 22/60 + 7/60^2 + 42/60^3 + 33/60^4 + 4/60^5 + 40/60^6$$

or

$$x = 1;22.07.42.33.04.40,$$

if written in the sexagesimal (base-sixty) notation using numbers from the decimal numeral system [10; p. 112, p. 114].

Sexagesimal notation was used during the Middle Ages in India, Arabic and Christian countries, and one would often see how Indian decimal numbers combined with fractions in sexagesimal form.

Value of the root x_F obtained by Fibonacci and correct value x_T in decimal system:

$x_F = 1.368\,808\,107\,853\,223\,5\ldots$ — Fibonacci's answer;
$x_T = 1.368\,808\,107\,821\,372\,6\ldots$ — true (correct) value.

If we insert the value x_F into the cubic equation, we will get a number slightly greater than 20:

$$20.000\,000\,000\,671\ \ldots,$$

that is, Fibonacci provided the answer with an accuracy up to nine decimal places (with a slight overestimation).

Amazing accuracy for that time!

[16] In Boncompagni's 1854 publication *Tre scritti inediti* … [8; p. 17] this is XXX, which is a mistake of the copyist; this copyist's mistake is also noted by Terquem [53; p. 115]. — *Footnote ours*.

However professor Ezra Brown and Jason Cornelius Brunson take notice of this overestimate [10]. They point out that in the times of Fibonacci there were known methods for performing calculations with a high accuracy and provide two such methods, noting that[17] [10; p. 118]:

> "There is no record that any other ways to find numerical approximations were available to mathematicians in the early thirteenth century, …"

One of the methods was definitely known to Fibonacci since chapter 13 of his manuscript *Liber Abaci* was dedicated to it. The chapter is entitled "*Incipit capitulum 13 de regulis elchatayn, qualiter per ipsam fere omnes questiones abaci soluuntur*" and begins as follows: "Elchataieym quidem arabice, latine duarum falsarum posicionum regula interpretatur, per quas fere omnium questionum solutio inuenitur; …" (The Arabic elchataym (*elchatayn; al-khata'ayn*) or the method of *Double False Position* in Latin is the method by which the solutions to nearly all problems are found; …) [7; vol. I, p. 318]. Already from the title of the chapter and the first sentence it becomes clear that the discussion is going to be about the method of the double false position.

It is not ruled out that the second method — the Ruffini-Horner method — was known to him [10; p. 118]:

> "While there is no direct evidence in any of Leonardo's writings that he even knew about the Ruffini-Horner method, it was well-known to the Islamic mathematicians of the day. As Calinger points out (see [**4**, p. 369]), Leonardo could easily have come across it during his travels."

Thereby, Fibonacci could provide an answer with high accuracy, probably spending only an hour or an hour and a half on calculations [10; p. 119].

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[17] Richard Maruszewski in the article *Fibonacci's Forgotten Number Revisited*, published in the journal *The College Mathematics Journal* (vol. 40, No 4, Sept 2009, pp. 248−251), points out that there was "another method, not mentioned by Brown and Brunson, but likely known to Fibonacci and more likely to be the one that he actually used."

But the last base 60 digit in his answer could not be $40/60^6$, since the methods available at that time could only yield a result with a certain underestimate, not an overestimate, so the last digit should have been $38/60^6$. In other words, Fibonacci should have indicated as the answer the number

$$x = 1;22.07.42.33.04.38,$$

which is a number with an underestimate, when compared to true value which is equal to 1;22.07.42.33.04.38.30.50… The result provided by him $x = 1;22.07.42.33.04.40$ is neither truncated, that is shortened by throwing away the last digits, nor rounded:

> "The two available methods yield underestimates, but the actual base-sixty expansion of the real root continues 1; 22,07,42,33,04,38,30,50, … . ⟨…⟩ The answer he gives is neither truncated nor rounded; why did he give it?" [10; pp. 118−119]

Why did Fibonacci indicate 40, and not 38? Authors list various causes, but the most probable, as it seems to them, is that he deliberately skewed the result so that nobody could guess by which method it was obtained. In the times of Fibonacci there was no tendency to disclose methods, as well as other secrets of the trade, since they "fed" their owners [10; p. 119], and the audience with the emperor was an excellent opportunity for Fibonacci to demonstrate his abilities, without revealing his methods.

A curious story!

The third [7; vol. II, pp. 234−236] of the problems mentioned by Fibonacci (see also [6; p. 7], [8; pp. 17−20]):

> "***De tribus hominibus pecuniam comunem habentibus.***
>
> Tres homines habebant pecuniam comunem, de qua medietas erat primi, tertia secundi. Sexta quoque pars tertij hominis; et cum eam in tutiori loco habere uoluissent, ex ea unusquisque cepit fortuitu; et cum totam ad tutiorem locum deportassent, primus, ex hoc quod cepit, posuit in comune medietatem, secundus tertiam, tertius sextam. Et cum ex hoc,

> quod in comune positum fuit, inter se equaliter diuisissent, suam unusquisque habuit portionem; queritur quanta fuit illa pecunia, et quot unusquisque ex ea cepit. Hec itaque questio, domine serenissime imperator, in palatio uestro pisis coram uestra maiestate, a magistro Iohanne panormitano mihi fuit proposita."
>
> (Three men possessed a pile of money, half of which belonged to the first, a third to the second, and a sixth to the third. Wishing to put it into a more secure place, each of them took a part at random to place the whole into that place, where the first placed half of what he took, the second one third, and the third one sixth. And when the total so returned was divided equally among them, each possessed what he was entitled to. How much money was there in total and how much did each man take? Such was the question, the Most Serene Lord Emperor, which Master John of Palermo proposed to me at the court of Your Majesty in Pisa.)

Fibonacci provides the complete solution and the answer; we provide the answer only [8; p. 19]:

> "… unde si ponatur rem[18] esse VII, tota pecunia erit XLVII, quia septuplum ipsius pecunie, scilicet de XLVII, equabitur XLVII rebus, scilicet multiplicationi de XLVII in VII. Nam septies XLVII sunt quantum XLVII vicibus VII, et quia primus cepit totam pecuniam minus duabus rebus, si de tota pecunia, que est XLVII, auferantur 2 res, scilicet XIIII, remanebunt XXX3 pro eo quod cepit primus homo. Item quia secundus cepit medietatem eiusdem pecunie minus una re et dimidia, si de medietate pecunie que est[19] XXIII $^{1}/_{1}$ auferatur res una et dimidia, scilicet X $^{1}/_{1}$, remanebunt XIII pro eo quod cepit secundus homo. Rursus quia tertius homo cepit quintam partem dicte pecunie minus re una et

[18] *rem* from *res* (Lat.) — thing; by this word Fibonacci denoted the unknown [43; p. 611]. — *Footnote ours.*

[19] In edition [8] by mistake it is indicated as XXXIII $^{1}/_{1}$. It should be XXIII $^{1}/_{1}$ — half of the overall sum XLVII [7; vol. II, p. 235]. Here XXIII $^{1}/_{1}$ is the 23½ [49; 3) Arithmetic & Number-Theoretic Recreations]. — *Footnote ours.*

quinta, si de quinta parte totius pecunie que est $^2/_5$ 9 auferatur res et quinta pars rei, scilicet $^2/_5$ 8, remanebit 1 pro eo quod cepit tertius homo.”

(… from here, if we assume one of the unknowns to be equal to 7, the total sum of money is going to be 47 (*unde si ponatur rem esse VII, tota pecunia erit XLVII*), … 33 is left — this is what the first man took (*remanebunt XXX3 pro eo quod cepit primus homo*). … the second man took (*cepit secundus homo*) 13, … the third man took (*cepit tertius homo*) 1.)

The solution to the problem with the use of the modern system of notation is provided in the Boncompagni’s work “Intorno ad alcune opere di Leonardo Pisano matematico del secolo decimoterzo” (*On Some Works of Leonardo of Pisa, Mathematician of the 13th Century*) [6; pp. 7−8] [20].

Similar analysis of and solutions to problems from *Flos*, *Liber Quadratorum* and *Epistola*, written in the modern mathematical notation can be found in the work of Angelo Genocchi[21] as well as in the works of Olry Terquem [53] and Ettore Picutti[22].

The excerpts from *Flos* are provided by us, firstly, so that the reader could see how mathematical treatises were stated in the times of Fibonacci. How was it possible to solve anything with such an imperfect form of notation! Secondly, these excepts show the real historic events that took place: the meeting with the Emperor and the challenge, arranged for Fibonacci.

The reader may try to solve the provided problems and evaluate their difficulty, and at the same time draw conclusions about the level of mathematical “tournament” organized in the far Middle Ages.

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[20] Before this reissue, the work *Intorno ad alcune* … was published in 1853 in the journal *Giornale Arcadico di Scienze, Lettere ed Arti*, volumes CXXXI, CXXXII and CXXXIII.

[21] Genocchi, Angelo. *Sopra tre scritti inediti di Leonardo Pisano pubblicati da Baldassarre Boncompagni: Note analitiche di Angelo Genocchi*. — Roma : Tipografia delle Belle Arti, 1855. — 126 p. (It.)

[22] Picutti, Ettore. *Il Flos di Leonardo Pisano : dal codice E.75 P. sup. della Biblioteca Ambrosiana di Milano*. Pp. 293−387 // Physis: Rivista Internazionale di Storia della Scienza. 1983, vol. 25, No 2. (It.)

The Latin texts from the Boncompagni's works are provided here as they are presented in the referred to sources: the same republished Boncompagni's texts may differ slightly in the placement of punctuation marks, absent at the time of Fibonacci, in corrections, although rare, certain words, and in mistakes, which in the subsequent publications were disposed of, though new ones crept it.

Lucas' proposition

In spite of the undeniable services to mankind, Fibonacci's name would be completely unknown to the general public if it were not for the presented by him, as a solution, numerical sequence in a problem about rabbits from his treatise *Liber Abaci*, and the proposition by French mathematician professor Edouard Lucas (François Édouard Anatole Lucas; 1842−1891) to name this sequence after Fibonacci.

And if the name Fibonacci is forever entered into the list of creators of mathematical science owing to his works, it was precisely owing to the problem about rabbits that he received wide recognition: few are those who have not heard about the Fibonacci numbers or the Fibonacci sequence.

Chapter III

LIBER ABACI

Handwritten treatise *Liber Abaci* — *The Book of Calculation* — appeared in 1202, and a second revised and expanded version of the manuscript appeared in 1228. Neither the first nor the second version of the original, written in Latin — the language of science at the time — has survived until our time. So modern publications are based on copies and translations of the second version of the manuscript.

In 1857 Boncompagni published the manuscript *Liber Abaci* [7; vol. I, pp. 1−459]. It was published in the language of the original — in Latin (more precisely, Fibonacci wrote in 13th century Tuscan Latin).

On the title page ("Il Liber Abbaci di Leonardo Pisano") Boncompagni indicated as the source the copy of Fibonaccci's manuscript from the central national library of the city of Florence[23]: "Codice Magliabechiano C. I, 2616, *Badia Fiorentina*, n.^{0} 73." Boncompagni describes this parchment manuscript among others in his publication "Della vita …" [5; p. 32] and notes it for its beautiful design ("Bellissimo Codice membranaceo in foglio …").

Liber Abaci begins thus [7; vol. I, p. 1], [44; p. 15]:

[23] Biblioteca Nazionale Centrale di Firenze; previously Biblioteca Magliabechiana; former private library of Antonio Magliabechi.

> Scripsistis mihi domine mi magister Michael Scotte, summe philosophe, vt librum de numero, quem dudum composui, uobis transcriberem: vnde uestrae obsecundans postulationi, ipsum subtiliori perscrutans[24] Indagine ad uestrum honorem et aliorum multorum utilitatem correxi. In cuius correctione quedam necessaria addidj, et quedam superflua resecaui.
>
> (You have written to me, Master Michael Scott, the greatest philosopher, about the book on numbers which some time ago I composed and transcribed to you; whence complying with your criticism, your more subtle examining circumspection, to the honor of you and many others I with advantage corrected this work. In this rectification new material has been added from which superfluous had been removed.)

Liber Abaci is a rather voluminous work: printed typographically by Boncompagni in 1857 the manuscript makes up the whole first volume "Scritti di Leonardo Pisano matematico del secolo decimoterzo" (*Writings of Leonardo of Pisa, Mathematician of the 13th Century*) and takes up 459 pages therein [7; vol. I], while the published translation into modern English by Laurence Sigler, runs to 636 pages [44].

Professor Laurence Sigler states the following about the 1857 Boncompagni's publication [44; p. 10−11]:

> "The Latin edition contains many misprints, mostly numerical ones, and itself notes several mistakes (sic) without the obvious correction to them, but there is not one case where the misprint or mistake causes an irresoluble ambiguity. Context is always sufficient to restore correct values. ⟨…⟩ There exist a number of manuscripts of *Liber abaci* in Europe which were examined by Boncompagni in preparing his definitive text. The Boncompagni text is complete and unambiguous."

The treatise acquaints the reader with place-valued decimal system and with Arabic numerals, and goes into detail to explain the rules

[24] Libri supposes that the word here should be *prescruptans* [5; p. 25], while in one of the manuscripts of *Liber Abaci* mentioned by Boncompagni the word *prescrutans* is used. [6; p. 90]. — *Footnote ours*.

we all now learn in school for adding, subtracting, multiplying and dividing numbers altogether with many problems to illustrate the rules.

As he points out in his biographical fragment, Fibonacci conceived his work for the Latin people, i.e. for the wide audience, which is why *Liber Abaci* contains a lot of material meant to help common people solve their everyday problems including those connected with business and commerce.

Extant manuscripts

Professor David Singmaster provides the following information on the Internet regarding the copies of manuscript *Liber Abaci* [49; 1) Sources in Recreational Mathematics: An Annotated Bibliography]:

> Richard E. Grimm … kindly gave me some details. There are 15 known MSS[25], all of the 1228 2nd ed. Six of these consist of 1½ to 3 chapters only; five of the others lack Chapter 10 and the second half of Chapter 9; one lacks Chapter 10 and one lacks much of Chapter 15, leaving two essentially complete texts. The last four MSS mentioned are the most important: Siena L.IV.20, c1275, lacking much of Chap. 15, "the oldest and best"; Siena L.IV.21, 1463 [Grimm said c1465 -- there are dates up through 1464 in interest calculations, but the Incipit specifically says 1463], which includes much other material from later writers, so it is at least double the size of L.IV.20; Vatican Palatino #1343, end of 13C, lacking Chap. 10; Florence Bibl. Naz. Conventi Soppressi C. 1. 2616, early 14C, "handsome but frequently badly faded" so "that a later hand found it necessary to rewrite what he saw there."

The information about *Liber Abaci* manuscripts and about their whereabouts can also be found in Lueneburg's book "Leonardi Pisani ..." [35; p. 19, pp. 315−316] and in Boncompagni's work

[25] MSS — manuscripts; MS — manuscript. — *Footnote ours*.

"Della vita …" [5]. In *Della vita* Boncompagni presents the facts known to him about Fibonacci and their interpretation by various authors and also provides detailed bibliographic descriptions of *Liber Abaci* manuscripts, discovered by him, and indicates the location of each one of them.

Liber Abaci or Liber Abbaci?

In the first volume of "Scritti di Leonardo Pisano …" [7; vol. I] Boncompagni on the title page, written in Italian, uses the word "*Abbaci*": "Il Liber Abbaci di Leonardo Pisano," while in the very beginning of the book — on the first page in the heading, written in Latin, the word "*Abaci*" is used: "Incipit liber Abaci Compositus a leonardo filio Bonacij Pisano." Both *Liber Abbaci* and *Liber Abaci* are acceptable and permissible, though "*Abaci*" is used more often today. It is rare to encounter the spelling "*Abbacci*" or "*Abacci*." Professor from Denmark, Jens Hoyrup (Høyrup), for example, in his publications uses the word "*abbaco*" or "*abbacus*" when speaking about mathematical tradition or *abbaco* culture and not about the calculating tool *abacus*:

> "In order to avoid confusion with the calculation-board or -frame I shall stick to the spelling *abbaco* and the Anglo-Latin analogue *abbacus* (except of course in quotations, which follow the source that is quoted)[26]."

Abbaco culture

Abbaco culture is a certain infrastructure which began to emerge in Italy in the time of Fibonacci in connection with the adoption of the positional decimal system: there are teachers (*maestri d'abaco*), who teach children and willing others counting and calculations in the new system; there are math textbooks (*libri d'abaco*), written in the students' native language, and not in Latin, as Fibonacci's writings; there are math schools *abbaco*,

[26] Footnote reference on p. 3 of the book: Høyrup, Jens. Jacopo da Firenze's *Tractatus Algorismi* and Early Italian Abbacus Culture. Basel : Birkhäuser, 2007. — 482 p. (Science Networks. Historical Studies. Volume 34). — *Footnote ours*.

where residents, mostly merchants, send their children of about ten years of age.

The first *abbaco* school in Bologna was mentioned in 1265, while in 1280 there were already schools in many cities of the northern Italy.

Since Fibonacci's manuscript *Liber Abaci* was the first known to us work on the decimal system of counting, written in Italy, there solidified a baseless belief that all the textbooks for *abbaco* schools took their beginnings from Leonardo Fibonacci's *Liber Abaci*, and the schools themselves owe their appearance to works of Fibonacci. As Hoyrup mentions in an annotation to his article "Leonardo Fibonacci and Abbaco Culture. A Proposal to Invert the Roles" [22]:

> "ABSTRACT. — Since long it has been regarded as an obvious fact in need of no argument that the mathematics of the Italian abbacus school was taken over from Leonardo Fibonacci's *Liber abbaci*."

And then [22; p. 24]:

> "As long as the existence of the late medieval and Renaissance Italian abbaco tradition has been recognized, it has been taken for granted by almost everybody that it had to descent from Leonardo Fibonacci's writings, at most with more or less marginal additions."

Translation of the manuscript's title

The title *Liber abaci* can be formally translated from Latin as *The Book of the Abacus* from *liber* — *book*, and *abaci* — genitive (or plural) of *abacus*. The abacus (Lat. *abacus*) is a calculating tool (counting device) used for many centuries — from approximately the fourth century B.C. — for performing arithmetic processes in Ancient Greece and Ancient Rome as well as in Asian and Arab world.

Based on the contents of the book and the modern language, in which the word *book* is usually not used in the titles, *Liber abaci* can be translated as *Arithmetic*.

Examples of such an interpretation are provided by Boncompagni:

> "Arismetrica Leonardi bigholli de pisis" [5; p. 30];
>
> "Leonardus Filius Bonaci Pisanus scripsit arithmeticam integram, in cuius calce est Algebra, anno 1202 quam postea correxit anno 1228, ..." [5; p. 209 (Sessione II)].

The title *The Book of the Abacus* means nothing to a modern reader, while the title *Arithmetic* is very far from the title of the original. What is more, not only did *Liber abaci* say nothing about using an abacus — it described methods that eliminated the need for such a device, thus the title *The Book of the Abacus* does not correspond to the contents and purpose of the book. Certain authors note this fact in their works.

Sigler, for example, writes [44; p. 4]:

> "… *Liber abaci* should not be translated as The Book of the Abacus. A *maestro d'abbaco* was a person who calculated directly with Hindu numerals without using the abacus, and *abaco* is the discipline of doing this."

Grimm notes [19; p. 101]:

> "… Leonardo uses the word <u>abbacus</u> for "calculation." By the twelfth century, in the latter part of which Leonardo was born, the older meaning of abacus as a calculation board had grown to include the operations which the abacus performed, namely calculation in general."

That is why we previously mentioned that we translate the title of the manuscript *Liber Abaci* as *The Book of the Abacus* purely formally.

As the correct titles for the manuscript *Liber Abaci* we can consider such as the *Book of Calculation* or the *Treatise on the Calculation*.

The Rabbit Problem

In *Liber Abaci*, among other things, the following problem is examined [7; vol. I, pp. 283−284]:

“***Quot paria coniculorum in uno anno ex uno pario germinentur.***

> Qvidam posuit unum par cuniculorum in quodam loco, qui erat undique pariete circundatus, ut sciret, quot ex eo paria germinarentur in uno anno: cum natura eorum sit per singulum mensem aliud par germinare; et in secundo mense ab eorum natiuitate germinant.”

In Sigler’s book “Fibonacci’s Liber Abaci …” the problem is presented in translation from Latin as follows [44; p. 404]:

“***How Many Pairs of Rabbits Are Created by One Pair in One Year***

> A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.”

This problem can also be found in the popular small book “Fibonacci Numbers” [55; pp. 1−2] by professor Nikolai Nikolaevich Vorobiev (1925−1996).

Perhaps this is one of the first mathematical models of population dynamics. The rabbits in this model are immortal and when a pair reaches the age of two months, it begets another pair of opposite sexes every month.

Chapter IV

FIBONACCI NUMBERS

Fibonacci presents the solution to the problem about rabbits in the form of the following sequence of numbers [7; vol. I, p. 284], [44; p. 404], [55; p. 2]:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377. \qquad (1)$$

These are the very same famous Fibonacci numbers. Fibonacci provides a way to get this result [7; vol. I, p. 284], [32; p. 5]:

> "Potes enim uidere in hac margine, qualiter hoc operati fuimus, scilicet quod iunximus primum numerum cum secundo, uidelicet 1 cum 2; et secundum cum tercio; et tercium cum quarto; et quartum cum quinto, et sic deinceps, donec iunximus decimum cum undecimo, uidelicet 144 cum 233; et habuimus suprascriptorum cuniculorum summam, uidelicet 377; et sic posses facere per ordinem de infinitis numeris mensibus."
>
> ("You can indeed see in the margin[27] how we operated, namely that we added the first number to the second, namely the 1 to the 2, and the second to the third, and the third to

[27] The sequence of type (1) is written in the margins of the manuscript page containing the problem about rabbits. The photocopy of this page from the mentioned manuscript C, I, 2616 is provided on the outside of the back cover of this book. — *Footnote ours.*

> the fourth, and the fourth to the fifth, and thus one after another until we added the tenth to the eleventh, namely the 144 to the 233, and we had the abovewritten sum of rabbits, namely 377, and thus you can in order find it for an unending number of months." [44; p. 405])

As we can see, Fibonacci numbers are the elements of a number sequence, formed by certain rules. This sequence possesses many interesting mathematical properties; some of which we will mention, but on the whole this is the subject of a separate discussion and there exists abundant literature on this topic.

Since a newborn pair becomes sexually mature in a month and after another month gives birth to a new pair of rabbits, the sequence (1) can be written as:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots \qquad (2)$$

This is the most common way of writing the Fibonacci sequence today.

In honor of the author of the problem, the sequence is called *Fibonacci sequence* (or *Fibonacci series*) and is denoted with a letter *F*, while its members (terms) — numbers that are included in the sequence — are called *Fibonacci numbers* [55; p. 3]. The sequence, as Fibonacci himself pointed out, is infinite.

From the sequence of Fibonacci numbers it can be seen how sharply and nonlinearly the number of individuals in the population increases.

Putting aside the living things, the sequence can be written as:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots \qquad (3)$$

This form, that is the sequence starting at zero, can be seen in commentary by Albert Girard (1595−1632) on page 170 of the first volume of works by Simon Stevin (1548−1620) [52]:

> "… progression 0, 1, 1, 2, 3, 5, 8, 13, 21, &c. dont chasque nõbre soit egal aux deux precedens, …"
>
> (… the progression 0, 1, 1, 2, 3, 5, 8, 13, 21, etc., every term of which is equal to the sum of two precedent numbers …)

The infinite sequence of Fibonacci numbers can be represented in the following form:

$$F_1,\ F_2,\ F_3,\ \ldots,\ F_n,\ \ldots$$

or

$$\{F_n\}.$$

Here the numbered terms F_1, F_2, F_3, …, F_n, … are members or elements of the sequence, lower indices (subscripts) 1, 2, 3, …, n number the elements of this sequence, F_n is the general or n-th term of the sequence. Thus, for example, F_{19} means the nineteenth Fibonacci's number.

The sequences of type (1), (2) and (3) are all one and the same: when moving from (1) to (2) and to (3) we simply add a new "previous" term to the left side such that the present numbers in the sequence do not change and so that the first two terms sum to third. If we continue to add terms to the left side, we obtain a sequence which extends in both directions from zero. An example of such a bilateral sequence, the so called *extended Fibonacci numbers*, can be found in the 1877 work of Edouard Lucas "Recherches sur plusieurs ouvrages de Léonard de Pise et sur diverses questions d'arithmétique supérieure" (*Research on several works of Leonardo of Pisa and on various questions of higher arithmetic*) [32; p. 10]:

"On a donc, en supposant la série continuée dans les deux sens

$$\ldots.\ u_{-3},\ u_{-2},\ u_{-1},\ u_0,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ \ldots.$$

$$2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ \ldots.\text{"} \qquad (4)$$

This sequence can be written in a general form as:

$$F_{-n},\ F_{-n+1},\ F_{-n+2},\ \ldots,\ F_{-1},\ F_0,\ F_1,\ \ldots,\ F_n$$

or

$$\{F_{\pm n}\}.$$

Here the lower indices of the sequence elements can take on integer values in the range from $-n$ to n.

The absolute values of the numbers from the *extended Fibonacci sequence* as can be seen from (4), are symmetrical about zero. In the region of negative values of index n ($n < 0$) the sign of the term F_{-n}, which is equal in absolute value to the respective term F_n from region $n > 0$, can be determined from the relation [55; p. 37]:

$$F_{-n} = (-1)^{n+1} F_n \tag{5}$$

Here the index n is the number of the element of the sequence in the region of positive index values.

For example, if $F_6 = 8$, then F_{-6} will be:

$$F_{-n} = F_{-6} = (-1)^{(6+1)} F_6 = (-1)^7 \cdot 8 = -8.$$

On the graph of Fig. 2 the positions of numbers of the extended Fibonacci sequence are designated with dotted markers, near which their values are indicated. The values n of the subscripts are laid out horizontally, and the values F_n of the numbers in the sequence are laid out on vertically.

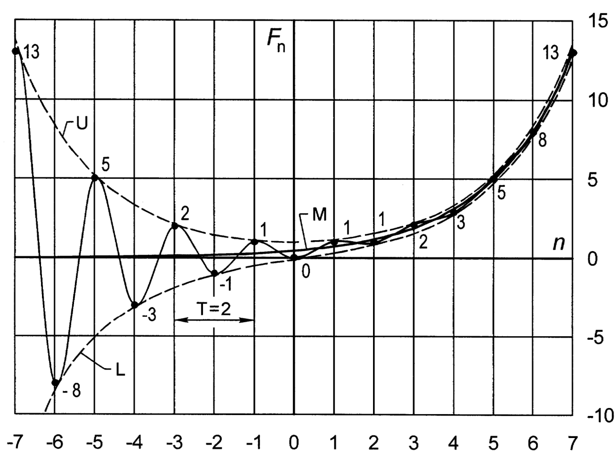

Fig. 2. Graphical representation of the extended classic Fibonacci sequence

If we connect the dots in Fig. 2 with a smooth curve we get a sinusoid. The period T of the oscillations of the sinusoid is equal to two units ($T = 2$).

The dashed curves U and L in graph of Fig. 2 are the upper and lower envelopes of the sinusoid. We can see that in the left part of the graph the amplitude of the oscillations of the sinusoid increases the further it is from the origin of the coordinates. The oscillations of the sinusoid occur about some mean value — the curve M on the graph. With increasing values of numbers in Fibonacci sequence the curve M asymptotically approaches the horizontal axis in the left part of the graph, while on the right — in the region of positive values of n — it goes to the right and upward into infinity.

It is known that the projection of the conic helix onto the plane is a sinusoid of varying amplitude. By analogy the sinusoid in the graph of Fig. 2 can be presented in the three-dimensional space as a spiral on the surface of a body resembling a curved horn, expanding like a funnel in the left side of the graph and narrowing in the right (Fig. 3). Fig. 3 is captured from such a viewpoint, i.e. the marks of Fibonacci numbers in space are located in such a way, so that the images in Fig. 2 and Fig. 3 look similar.

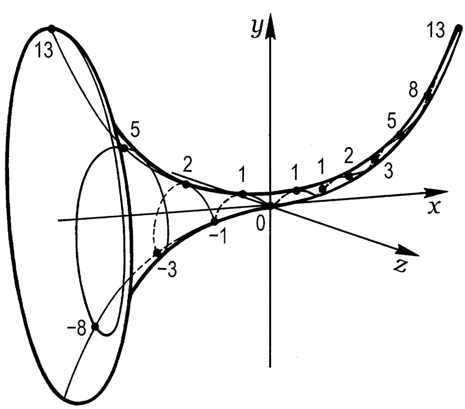

Fig. 3. Fibonacci numbers in the three-dimensional space — the three-dimensional Fibonacci spiral

Is there any sense in such three dimensional presentation? But it looks nice.

Fibonacci sequence of type (3) can be formed with the help of the following relation:

$$F_n := F(n) := \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{(n-1)} + F_{(n-2)} & \text{if } n > 1. \end{cases} \tag{6}$$

Here n — the number of the term in the sequence — positive integer or zero;

F_n — value of the n-th term of the sequence $\{Fn\}$;

$F(n)$ — value of the n-th term of sequence $\{Fn\}$ as a function of its number — lower index n.

In other words, if we sequentially find all Fibonacci numbers by using the expression (6) then in the end we will get Fibonacci sequence of type (3).

Expression (6) means the following: a particular Fibonacci number F is equal by definition to the value, specified by the conditions of the right side of the expression. Conditions 0 for $n = 0$ and 1 for $n = 1$ in expression (6) are actually the starting conditions $F_0 = 0$ and $F_1 = 1$ which in the end bring us to sequence of type (3).

The Fibonacci sequence can be built from left to right or right to left or simultaneously in both directions. The sequence built from left to right, that is when the resultant term of the sequence is located in the Fig. 2 to the right of the preceding ones, can be obtained with the help of the following simple recurrence relation[28] [55; p. 2]:

$$F_n = F_{n-1} + F_{n-2}\,. \tag{7'}$$

[28] A recurrence relation (from Latin *recurro*; *recurrens*, *recurrentis* — returning, recurrent) is an equation that defines a sequence recursively: the n-th term of the sequence is defined as a function of the preceding terms.

The formula (7') can look slightly different, which, however, does not change its meaning; for example, we can write:

$$F_{n+1} = F_n + F_{n-1} \quad \text{or} \quad F_{n+2} = F_{n+1} + F_n. \tag{7''}$$

For a sequence built up from right to left, as follows from (7'):

$$F_{n-2} = F_n - F_{n-1} \tag{8'}$$

or the same as follows from (7''):

$$F_{n-1} = F_{n+1} - F_n; \qquad F_n = F_{n+2} - F_{n+1}. \tag{8''}$$

In the formulas (7'), (7''), (8') and (8''), i.e. in formulas of type (7) and (8), F_n is the n-th term of the Fibonacci sequence and n is a positive or negative integer, or zero.

Formulas of type (7) and (8) are more compact than of type (6), however in order to write the sequence with their help it is necessary to provide the starting conditions. One needs to specify the values of the first two sequence members by picking for them two consequent numbers from the sequence of type (4). So if we denote the first number of the sequence as F_1, the second as F_2, and we let $F_1 = 1$ and $F_2 = 2$, then the value of the third term (number) F_3, according to (7') is:

$$F_3 = \boxed{F_n = F_{n-1} + F_{n-2}} = F_{3-1} + F_{3-2} = F_2 + F_1 = 2 + 1 = 3$$

(here $n = 3$);

or (according to 7'') is:

$$F_3 = F_{1+2} = \boxed{F_{n+2} = F_{n+1} + F_n} = F_{1+1} + F_1 = F_2 + F_1 = 2 + 1 = 3$$

(here $n = 1$).

As a result we get the sequence of type (1). If we denote the first two terms of the sequence as F_0 and F_1 and let $F_0 = 0$ and $F_1 = 1$, then we will get the sequence of type (3).

Similarly, with the help of the formulas of type (8) we get a sequence that is built in the opposite direction. By using relations of type (7) and (8) we can create a sequence $\{F_{\pm n}\}$.

The sequence formed with the help of the formulas (7) or (8) is the simplest second-order linear recurrence equation without coefficients where to find the next term we must know the two that precede it.

The recurrence relation and Albert Girard

The literature often indicates that the recurrence relation of type (7) was first written down by French mathematician Albert Girard in his commentary on pages 169−170 of the first volume of Simon Stevin's works[29] [52]. For example, professor Leonard Eugene Dickson (1874−1954), when writing about the recurring series in his widely known fundamental work "History of the Theory of Numbers," mentions Fibonacci sequence of type (1), the sequence of type (3) and adds the following [13; p. 393]:

> "Albert Girard noted the law $u_{n+2} = u_{n+1} + u_n$ for these series."

Dickson refers to the Albert Girard's commentary on page 169 in the first volume of the Simon Stevin's collected works [52].

Commentary by Albert Girard

The indicated above commentary is referred to and partly provided by Edouard Lucas in his work "Recherches sur plusieurs ouvrages de Léonard de Pise ..." [32; pp. 6−7].

There exists a paper by professor of mathematics Robert Simson (1687−1768) entitled "An Explication of an Obscure Passage in Albert Girard's Commentary upon Simon Stevin's Works" [46], dedicated to the explanation of obscure places in Girard's commentary on pages 169−170.

In 1902 appears Georges Maupin's (1867−?) second book "Opinions et curiosités touchant la mathématique" (*Opinions and*

[29] Simon Stevin's works were translated into French by Albert Girard and published in 1634, two years after Girard's death.

curiosities in mathematics) where he provides the commentary by Albert Girard in full, substituting a more modern font and removing misprints, and provides his own commentary on this commentary [38; pp. 203−209].

However neither Lucas, nor Simson, nor Maupin mention the formula provided by Dickson. It is not present in this, or any other Girard's commentaries in the mentioned collected works. What is more, the formula could not have existed in the form as provided by Dickson — Girard used a system of mathematical notations different from today's.

So it is necessary to correct Dickson's information. There is an electronic version of the publication mentioned by Dickson [52] on the Internet, where one can see the page 169 with the commentary. For example, by URL:

http://diglib.hab.de/drucke/n-11-2f-helmst/start.htm

Generalizations of Fibonacci numbers

We can form infinitely many different numerical sequences satisfying the condition (7), if as the initial we choose two arbitrary numbers, for example [55; p. 3]:

$$2,\ 5,\ 7,\ 12,\ 19,\ 31,\ 50,\ \ldots, \tag{9}$$

$$1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ \ldots, \tag{10}$$

$$-1,\ -5,\ -6,\ -11,\ -17,\ \ldots \tag{11}$$

Can we consider sequences (9), (10) and (10) to be Fibonacci sequences? It turns out that we can. There exists numerous modifications or the generalizations of Fibonacci numbers, where the provided above "classic" Fibonacci sequence — sequences (1), (2) and (3) — is just a special case of the family of recurrent numerical sequences, expressed by a more general formula.

So the point of view of Brother Alfred[30] about the definition of the Fibonacci sequence is the following [9; p. 82]:

[30] Brother U. Alfred (Alfred Brousseau; 1907-1988) American monk, photographer and mathematician. While teaching at St. Mary's College, Calif., Brother Alfred also continued his own studies in physics, and in 1937 he received

“Recurrent sequences in which each term is the sum of the two preceding terms are known as Fibonacci sequences. The law of recurrence for all such sequences is

$$\mathrm{T}_{\mathrm{n}+1} = \mathrm{T}_{\mathrm{n}} + \mathrm{T}_{\mathrm{n}-1}.$$

Starting with the values of T_1 and T_2, it is possible to build up such a sequence. Thus, if $\mathrm{T}_1 = 3$ and $\mathrm{T}_2 = 11$, it follows that $\mathrm{T}_3 = 14$, $\mathrm{T}_4 = 25$, $\mathrm{T}_5 = 39, \cdots$.

One can go on to variations of this idea. For example:

$$\mathrm{T}_{\mathrm{n}+1} = 2\mathrm{T}_{\mathrm{n}} + 3\mathrm{T}_{\mathrm{n}-1}$$

or

$$\mathrm{T}_{\mathrm{n}+1} = \mathrm{T}_{\mathrm{n}} + \mathrm{T}_{\mathrm{n}-1} + \mathrm{T}_{\mathrm{n}-2}.\text{”}$$

As we can see from the expression $\mathrm{T}_{\mathrm{n}+1} = \mathrm{T}_{\mathrm{n}} + \mathrm{T}_{\mathrm{n}-1}$ provided by Brother Alfred, in the simplest case the generalization consists of using any two numbers as initial values in recurrence relation (7) or (8), and not just the two consecutive numbers from the classic Fibonacci sequence.

The expression by which one can obtain a generalized Fibonacci sequence $\{G_n\}$ can be written, for example, in the following way:

$$G_n = G_{n-1} + G_{n-2}, \tag{12}$$

where G_{n-1} and G_{n-2} are any two numbers not both zero.

From here it directly follows that the classical Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ can be considered special cases of generalized Fibonacci sequence $\{G_n\}$:

$$\{F_n\} \subset \{G_n\}; \ \ \{L_n\} \subset \{G_n\}.$$

---

the Ph.D. degree from the University of California. Co-founder of “The Fibonacci Association.” *The Fibonacci Association* was founded in December, 1962 and incorporated in 1963. Since 1963 association publishes the journal “The Fibonacci Quarterly.” The journal can be considered a trendsetter in everything to do with Fibonacci numbers.

Lucas sequence is denoted with the letter L in honor of French mathematician Edouard Lucas and it results from swapping the first two terms in the sequence (1):

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots$$

The Twin Shining Stars

> "The Fibonacci sequence and the Lucas sequence are the two shining stars in the vast array of integer sequences. They have fascinated both amateurs and professional mathematicians for centuries, and they continue to charm us with their beauty, their abundant applications, and their ubiquitous habit of occurring in totally surprising and unrelated places."

This is how professor Thomas Koshy refers to these two sequences in his book "Fibonacci and Lucas Numbers with Applications" [26; p. xi].

Further methods of generalization

From what Brother Alfred says, further methods of generalizations depend on the imagination of the author and can follow, for example, the path of introducing coefficients in front of terms of the sequence — this is the second of his provided examples $T_{n+1} = 2T_n + 3T_{n-1}$ or by changing the number of summed terms which is his third example $T_{n+1} = T_n + T_{n-1} + T_{n-2}$.

A generalized Fibonacci sequence, for example, in the article "A generalization of the connection …" by Joseph A. Raab is presented as follows [41; p. 23]:

$$u_n = au_{n-1} + bu_{n-2}, \tag{13}$$

where both of the initial sequence terms are arbitrary numbers not simultaneously zero, and a and b are real numbers.

As we can see, with the specified methods of generalization (12) and (13), any recurrence sequence of second order, including, for example, such as (9), (10) or (11), is already considered a Fibonacci sequence.

Presented below are some methods of obtaining the generalized Fibonacci sequences along with examples of families of recurrence sequences.

Examples of generalizations

If the following term is formed by summation of the preceding term with itself, we obtain a (binary) sequence of type:

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots \tag{14}$$

or

$$2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ \ldots,\ 2^N,\ \ldots$$

If the following (fourth) term is formed not by the sum of the two consecutive preceding terms, but by the sum of the first and third preceding terms we obtain the sequence of type:

$$1,\ 1,\ 1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ \ldots \tag{15}$$

Sequence (15) shows, in particular, how the total number of cows increases in the problem by Indian mathematician Narayana Pandit or Pandita (1340−1400). This problem is similar to the Fibonacci's rabbits problem, and Narayana Pandit provides it in 1356 in his work, the title of which after transliteration of Indian writing by Roman characters can be written as "Ganita Kaumudi" [47; p. 619, p. 621−622]:

> "A cow gives birth to one calf every year. The calves become young and themselves begin giving birth to calves when they are three years old. O learned man, tell me the number of progeny produced during twenty years by one cow."

The solution method and answer (2 745 progenies) are provided.

Incidentally, is the author not too young for his work? Taking into account that the dating of Narayana's work is more reliable then that of his life period, it is probably necessary to shift the latter by 10−20 years toward the earlier dates.

The difference from the Fibonacci's problem in the example provided here is that rabbits require two steps to produce the first offspring (one month to become mature and another to breed) while cows need three (1 year + 1 year + 1 year). Therefore in the cows problem the sequence of type (15) has three ones in the beginning, and not two as in the rabbits sequence of type (2).

The recurrence relation for the sequence of type (15) can be represented as follows:

$$F_n = F_{n-1} + F_{n-3}.$$

If we form the following (fifth) term by the sum of the first and the fourth preceding terms, then we obtain a sequence of type:

$$1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, \ldots \tag{16}$$

The recurrent relation in this case can be written as follows:

$$F_n = F_{n-1} + F_{n-4}.$$

By increasing the distance between the two initial terms we obtain more new sequences. Since our series is formed by varying the distance between the summed terms, we can introduce the following parameter into formula (7):

$$F_n = F_{n-1} + F_{n-1-k} \tag{17}$$

or

$$F_n = F_{n-1} + F_{n-k-1}.$$

Here, an integer $k = 0, 1, 2, 3, \ldots$ determines the distance between summed terms.

As we can see, the sequences of type (14), (15) or (16) are completely unlike the classic Fibonacci sequence. When $k = 1$, formula (17) turns into (7'). From (17) the classic sequence can be obtained as a special case with the appropriate initial conditions when $k = 1$. The appropriate initial conditions can be as, for example, in (6).

The sequence of type (14) is obtained if we let $k = 0$ in (17) and if the first term of the sequence is equal to one.

Tribonacci, Tetranacci

It is possible to obtain a family of recurrent sequences by changing not the distance between two summed terms, but the number of summed terms themselves — as in the third of the Brother Alfred's examples provided above. Thus, if instead of two terms we add together three, we get "tribonacci sequence" [16] or "tribonacci numbers":

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \ldots$$

The recurrence relation for the provided sequence can be written as follows:

$$F_n = F_{n-1} + F_{n-2} + F_{n-3}.$$

Similarly, by adding together four preceding terms we obtain the Tetranacci or Tetrabonacci sequence [16]:

$$0, 0, 0, 1, 1, 2, 4, 8\ 15, 29, 56, 108, 208, 401, 773, 1490, \ldots$$

And continuing in the same way we obtain the following types of sequences: pentanacci (pentabonacci), hexanacci (hexabonacci), heptanacci (heptabonacci) — k-nacci in the general case. As a result we have the family of recurrent sequences which differ in the number of summed terms. The classic Fibonacci sequence is obtained as a special case when the number of summed terms is equal to two and the initial conditions are as in (6), for example.

There exists a vast literature on all possible ways of forming generalized Fibonacci sequences.

The Fibonacci sequence attracts the attention of mathematicians as well as scientists from other fields, philosophers and representatives from the world of art, since it possesses, as we already mentioned, a number of rather interesting properties and patterns. And the number of these properties, regularities, occurrences and applications is such, that since 1963 *The Fibonacci Association* publishes a journal titled *The Fibonacci Quarterly* with material relating to Fibonacci sequences.

Binet's formula and the curve M

If it is necessary to find the value of an arbitrarily large number of classic Fibonacci sequence, the calculations using the recurrent formulas (7') or (8') will require a lot of steps. This can be avoided with the help of Binet's formula:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{18}$$

Binet's formula expresses F_n as a function of the n-th number and allows one to find the value of any number from the classic Fibonacci sequence without the use of recurrent relation for calculating sequentially all the numbers up to the one required.

In the region of positive values of the argument n, i.e. for Fibonacci numbers on the right side of the graph on Fig. 2 with $n \geq 0$ one can use the simplified formula:

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{\varphi^n}{\sqrt{5}}, \tag{19}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

The result of the calculation is then rounded to the nearest integer.

By using formula (19) for calculations, we are in effect finding the values of the points belonging to a curve of type

$$y = k\varphi^x, \tag{20}$$

where the value of the coefficient k, as can be seen from (19), equals

$$k = \frac{1}{\sqrt{5}}.$$

By rounding the result to the nearest integer, which then is a Fibonacci number, we move "away" from the curve. Thus the discretization and the approximation of the curve with integers (when n is an integer) gives us the Fibonacci sequence. The curve of type (20) with the given k is exactly the curve M in Fig. 2. Hence the equation of curve M is:

$$y = \frac{1}{\sqrt{5}}\varphi^x = \frac{\varphi^x}{\sqrt{5}}, \tag{21}$$

or with up to two significant digits:

$$y = 0.45\varphi^x.$$

From Fig. 2 it is clear why it is only possible to use formula (19) for finding the Fibonacci numbers on the right side of the graph (in the first quadrant): Fibonacci numbers in the second and third quadrants on the left side of the graph of Fig. 2 diverge from curve M, which asymptotically approaches the x-axis.

The simplified formula for calculating the values of the Fibonacci numbers in the region of negative values of argument n ($n \leq 0$) — on the left side of Fig. 2 — makes use of only the second term in the square brackets of Binet's formula (18):

$$F_n \approx -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

One of the properties of Fibonacci sequence

One of the properties of Fibonacci sequence is that with an increase in the absolute value of sequence members, the ratio between the following term G_{n+1} and the preceding G_n in the limit is equal to φ:

$$\varphi = \lim_{n\to\infty}\frac{G_{n+1}}{G_n} = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Why in the limit only

From Fig. 2, using classic Fibonacci sequence as an example, we can see why it is only in the limit that the ratio of two adjacent Fibonacci numbers is equal to φ. The Fibonacci numbers, as can be seen from the graph, perform “damped oscillations” about the smooth curve M as it goes into the upper right to infinity on the right side of the graph. In other words, as the values increase, Fibonacci numbers approach the curve, which in the limit results in φ for adjacent numbers, since for curve M the ratio of values for two consecutive numbers y_{n+1} and y_n for values of argument x which differ by one, is equal to φ:

$$\frac{y_{n+1}}{y_n} = \frac{\varphi^{x+1}}{\sqrt{5}} : \frac{\varphi^x}{\sqrt{5}} = \frac{\varphi^x \cdot \varphi}{\sqrt{5}} : \frac{\varphi^x}{\sqrt{5}} = \varphi .$$

The numbers on the left side of the graph in Fig. 2 are mirror image (in absolute value) of the numbers on the right side of the graph, so the assertion also holds for them that in the limit the ratio of two adjacent Fibonacci numbers is equal to φ (in absolute value!); however since the adjacent Fibonacci numbers on the left side of the graph in Fig. 2 are always of the opposite magnitudes, their ratio in the limit approaches $-\varphi$:

$$\lim_{n\to\infty} \frac{-G_{n+1}}{G_n} = -\varphi \quad \text{or} \quad \lim_{n\to\infty} \frac{G_{n+1}}{-G_n} = -\varphi .$$

Here the negative Fibonacci number $-G_{n+1}$ is greater in absolute value than the adjacent to it G_n and is placed further than G_n from the origin on the left side of the graph in Fig. 2. Similarly the observation is also valid with relation to the pair of numbers G_{n+1} and $-G_n$.

The ratio $\varphi \approx 1.618$ is best known as the *golden ratio* or the *golden section*.

Chapter V

GOLDEN SECTION

The golden section (Latin: *sectio aurea*) or the golden ratio, or the golden cut, or the golden proportion, or the golden relation, or the golden coefficient or the divine proportion is a specific ratio of numbers or values. This relation is an irrational (infinite) number and is equal to

$$\varphi = 1.618\,033\,988\,749\,894\,848\,\ldots$$

or

$$\varphi = \frac{1+\sqrt{5}}{2}. \tag{22}$$

It is believed that the Greek letter φ (phi) to denote this relation was proposed in approximately 1909 by American mathematician Mark Barr (? −1950). Sir Theodore Andrea Cook (1867 −1928) reports this in his book "The Curves of Life" as follows [12; p. 420]:

> "Mr. Mark Barr suggested to Mr. Schooling that this ratio should be called the ϕ proportion for reasons given below. Adopting this symbol for the common ratio the equation becomes:
>
> $$\phi^n = \phi^{n-1} + \phi^{n-2} \qquad \phi^2 = \phi + 1 \qquad \phi = (1 \pm \sqrt{5}) \div 2\,.$$

⟨…⟩ The symbol ϕ given to this proportion was chosen partly because it has a familiar sound to those who wrestle constantly with π (the ratio of the circumference of a circle to its diameter), and partly because it is the first letter of the name of Pheidias, in whose sculpture this proportion is seen to prevail when the distances between salient points are measured."

As we can see Barr meant the notation $\phi = (1 \pm \sqrt{5}) \div 2$. William Schooling writes the following about φ [12; pp. 441–442]:

> "The ϕ ratio has long been known, but it receives an added significance when it is recognised as the common ratio of a geometrical progression, in which the sum of any two consecutive terms equals the next term. The progression is:
>
> $$1,\ \phi,\ \phi^2,\ \phi^3 \ldots\ \phi^n,\ \phi^{n+1},\ \phi^{n+2},\ \text{etc.}$$
>
> Then, $\quad \phi^n + \phi^{n+1} = \phi^{n+2}$
>
> Divide by $\phi^n \qquad 1 + \phi = \phi^2 \qquad \phi = (1 \pm \sqrt{5}) \div 2$
>
> … and ϕ may have four values, *i.e.*, —
>
> $+\ 1\dot{}618$, etc. $\quad -\ 1\dot{}618$, etc. $\quad +\ \dot{}618$, etc. $\quad -\ \dot{}618$, etc.
>
> Writing in the *Daily Telegraph* for January 21st, 1911, I said there is a "very wonderful number which may be called by the Greek letter phi, of which nobody has heard much as yet, ⟨…⟩." I will now turn to the famous Fibonacci series and compare it with the ϕ progression, taking the value of ϕ as $1\dot{}618$, etc."

Common today is $\varphi = (1 + \sqrt{5}) \div 2$ and not $\varphi = (1 \pm \sqrt{5}) \div 2$. The notation φ is not strictly required. The uppercase Φ denotes the inverse value of φ:

$$\Phi = 1/\varphi = 0.6180\ldots$$

The direct and inverse values are related by:

$$\Phi = \varphi - 1,$$

that is Φ can be obtained both by division of one by φ, and by subtraction of one from the coefficient φ. Unique property!

It is also interesting that

$$\Phi + \varphi = \sqrt{5}.$$

Division of a line segment

The problem of division of a line segment into its extreme and mean ratio is a classic example of the golden section. One must cut a segment into two in such a way that the length of the whole segment — *AB* in Fig. 4 — is to the longer segment *CB* as the longer segment *CB* is to the shorter segment *AC*.

In other words we have the condition $AB : CB = CB : AC$ and we must find the position of the point *C*.

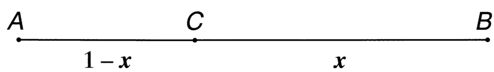

Fig. 4 Division of a segment into its extreme and mean ratio

Let $CB = x$. Then for the segment *AB* of unit length ($AB = 1$) the length of its other part will equal:

$$AC = AB - CB = 1 - \text{x},$$

and the above condition $AB : CB = CB : AC$ will look as follows:

$$1 : x = x : (1 - x)$$

or

$$x^2 + x - 1 = 0.$$

The positive root x_1 of this quadratic equation (the length x of segment CB) is

$$x_1 = \frac{\sqrt{5} - 1}{2} \approx 0.6180.$$

Let us find ratios $AB : CB$ and $CB : AC$:

$$\frac{AB}{CB} = \frac{1}{x} = \frac{1}{\frac{\sqrt{5}-1}{2}} = \frac{1+\sqrt{5}}{2} = \varphi ,$$

$$\frac{CB}{AC} = \frac{x}{1-x} = \frac{\frac{\sqrt{5}-1}{2}}{1-\frac{\sqrt{5}-1}{2}} = \frac{1+\sqrt{5}}{2} = \varphi .$$

As we can see the division of a line segment into its extreme and mean ratio gives the ratio of its parts which is the golden section φ.

It may seem surprising that there is a connection between Fibonacci's rabbits and the line segment. However there is nothing surprising about it: the following term in the Fibonacci sequence, as the line segment *AB*, consists of two parts — the two preceding terms which sum just as the line segments *AC* and *CB*.

What is surprising is that not only for classic Fibonacci sequence but for any another, formed by the rule of "following term is equal to the sum of the two preceding it," regardless of the first two initial terms — integers or fractions, rational or irrational — as long as both are not zero, the ratio of the two adjacent numbers u_n+1 and u_n as they move away from the start of the sequence converges to the point of the golden section φ:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \varphi .$$

We can ascertain this by checking the recurrent sequences provided earlier.

The fact, that the limit is equal to φ for any initial numbers (if both not zero), means that the specific limit value is a property of the equation and not of some particular numbers. So, from this point of view, one should not deify the classic Fibonacci sequence, by assigning only to it the unique property of the ratio between adjacent terms of the sequence. In this case the ratio, equal in the limit to φ, is obtained from the recurrent sequence of the second order of type (12) since this is its property. For other recurrent sequences the values of the limit will be different.

Let us give a few examples showing to which limit the ratio of the adjacent terms tends for some types of recurrent sequences:

$k_1 = 2$ — for binary sequence of type (14): $F_n = F_{n-1} + F_{n-1}$;
$k_2 \approx 1.928$ — for sequence $F_n = F_{n-1} + F_{n-2} + F_{n-3} + F_{n-4}$;
$k_3 \approx 1.839$ — for tribonacci sequence $F_n = F_{n-1} + F_{n-2} + F_{n-3}$;
$\varphi \approx 1.618$ — for (12) and the classic $F_n = F_{n-1} + F_{n-2}$;
$k_4 \approx 1.466$ — for sequence $F_n = F_{n-1} + F_{n-3}$;
$k_5 \approx 1.380$ — for sequence $F_n = F_{n-1} + F_{n-4}$;
$k_6 \approx 1.256$ — for sequence $F_n = F_{n-1} + F_{n-7}$;
$k_7 \approx 1.237$ — for sequence $F_n = F_{n-2} + F_{n-5}$.

On the division in extreme and mean ratio one can refer to the well written and very informative book "A Mathematical History of Division in Extreme and Mean Ratio" by Roger Herz-Fischler [21].

Kepler on the two treasures of geometry

When writing about golden section or about Fibonacci, authors quite often provide the following statement on behalf of the known astronomer and mathematician Johannes Kepler (1571−1630) in which he expresses his admiration of the golden ratio [18; p. 243], [26; p. 242]:

> "Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name a precious jewel."

The quote about the two treasures is provided more often than not without an indication as to where Kepler said it, so it is very difficult to find a reference to this saying — see for example the quotation without a reference on page 58 of Carl Boyer's book "A History of mathematics"[31] or on page 23 in "The Divine Proportion" by Huntley[32], or in the article by Martin Gardner (1914−2010) "The Cult of the Golden Ratio" [18; p. 243].

The earliest passage we could find with a reference to the source is the following quote in German in the text "Der goldne Schnitt. Beitrag zur Geschichte der Mathematik und ihrer Anwendung " (*The Golden Section: To the History of Mathematics and its Application*) by Ludwig Sonnenburg (1820−1888) in 1880−1881 Program of the Royal Gymnasium in Bonn [51; p. 9]:

> "Die Geometrie hat zwei grosse Schätze, einer ist der Satz des Pythagoras, der andre die Teilung einer Linie im äussern und mittlern Verhältnis, den erstern kann man einer Masse Goldes vergleichen, den andern kann man einen kostbaren Edelstein nennen."
>
> (Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name a precious jewel.)

As a source Sonnenburg indicated [51; p. 9, p. 3] the pages 140 and 145 of Kepler's "Prodromus dissertationum cosmographicarum, continens Mysterium cosmographicum de admirabili proportione …" [17; p. XV; pp. 95−187] from the first volume of an eight-volume work by Christian von Frisch (1807−1881) "Joannis Kepleri astronomi opera omnia" (*The Complete Works by the Astronomer Johannes Kepler*) [17].

Because of the long title the work is usually contracted to "Mysterium Cosmographicum"[33] (*The Cosmographic Mystery* or

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[31] Boyer, Carl B. *A History of mathematics*. — 2nd. ed. — New York : John Wiley & Sons, 1989. — 762 p.

[32] Huntley, H. E. *The Divine Proportion: a Study in Mathematical beauty*. — New York : Dover Publications, 1970. — 186 p.

[33] Professor Roger Herz-Fischler, for example, refers to *Mysterium Cosmographicum* on page 175 of his book *A Mathematical History* … [21] when translating to English the Kepler's statement about the two treasures of geometry as it is originally presented by Kepler. The reference to *Mysterium Cosmographicum*,

The Secret of the World). Christian von Frisch himself published this work under the title "Prodromus dissertationum Cosmographicarum seu Mysterium Cosmographicum" [17; p. 671].

On page 140 (with an endnote **r** on page 145) suitable for our case is the following:

> "… duos nempe esse geometriae thesauros, unum: subtensae in rectangulo rationem ad latera, alterum: lineam extrema et media ratione sectam,[r]) …"
>
> (… geometry no doubt has two great treasures: one, the ratio of the hypotenuse in a right-angled triangle to the sides, and the other, the line divided in the mean and extreme ratio.[r)] …)

The endnote **r** on page 145:

> " r) Duo theoremata infinitae utilitatis, eoque pretiosissima, sed magnum discrimen tamen est inter utrumque. Nam prius, quod latera recti anguli possint tantum, quantum subtensa recto, hoc inquam recte comparaveris massae auri: alterum, de sectione proportionali, gemmam dixeris."
>
> (**r)** They are two theorems of infinite usefulness, and so of the greatest value; however there is a great difference between the two. For the former — that the squares of the sides of a right triangle are equal to the square of the hypotenuse — that, I say, can rightly be compared to a gold nugget; the second, on proportional division, can be called a gemstone.)

As we can see the name Pythagoras is not mentioned here at all. It is present on page 148 not mentioned by Sonnenburg:

> "Atque hactenus usui fuit aureum illud theorema Pythagorae de potentiis laterum in triangulo rectangulo. In ceteris duobus corporibus altero illo geometriae thesauro opus est, de linea secundum extremam et mediam rationem secta…" [17; p.148]

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without specifying the location of the statement is also present in Koshy's work [26; p. 242], however the text is similar to the one provided by Sonnenburg.

(So far we have been able to use the golden theorem of Pythagoras on the squares of the sides in a right-angled triangle. For the other two solids we need the other treasure of geometry — the line divided in the extreme and mean proportion …)

As we can see the ideas expressed by Kepler in the quoted passages correspond to the content of the phrase by Sonnenburg, but that phrase itself is not present in Kepler's work. So we must either find where Kepler says it exactly in this way — for example, there is no such phrase in the Kepler's works "A New Year's Gift or On the Six-Cornered Snowflake" (*Strena seu de Nive Sexangula*) [24] or "The Harmony of the World" (*Harmonices Mundi*), — or assume that the phrase was attributed to him — in this case most likely by Sonnenburg, since he provides it in quotes as a citation, even though Kepler wrote no such thing.

pro et contra (the pros and cons)

An enormous amount of literature has been dedicated to the golden ratio (golden section). Some elevate it to cult status and believe that the golden ratio is encountered at every step and plays an important role in the universe, while others consider that all this, aside from the purely mathematical properties of the golden ratio, is far-fetched and is either intentionally misinterpreted or an honest mistake.

In publications by the proponents of the idea that the golden ratio is ubiquitous (we shall call them "proponents" for short) it is often reported that the golden ratio is as old as nature itself. It is a proportion used by nature to give form to the sea shell and the leaves and crowns of trees, insects and people, atoms and molecules of DNA, hurricanes and galaxies. People have long used the golden ratio to create the most beautiful and attractive works of art and architecture. The proportions of the golden ratio are pleasing to the eye and object is perceived aesthetically better if the ratios of its proportions are close to 1.618. The new examples of the manifestation of the golden ratio in the real world are periodically discovered and provided, and some researchers from different fields are inclined to think of it as being one of the fundamental constants of the living and inanimate nature.

Those who do not agree with the assertions of the proponents or doubt their truth (we shall call them "skeptics" for short in the future) say that there is an ordinary cult of the golden ratio, when a certain phenomenon is given special meaning, which in reality is not there. The elevation of the golden ratio to cult status became possible because people believe in what is written and do not check that which is presented to them, that is, they accept the given uncritically. In reality there are no advantages of the golden ratio over the other coefficients, and the golden ratio φ is encountered in the world around us no more than the other coefficients.

Skeptics point out that the most widespread misconceptions about golden section, migrating from one publication to another, are the following:

- the term "golden section" was used in antiquity;
- the Great Pyramid was designed to conform to φ;
- the Greeks used φ in the Parthenon;
- a golden rectangle[34] is the most aesthetically pleasing rectangle;
- many painters, including Leonardo da Vinci, used φ;
- the human body exhibits φ;
- the shell spiral of Chambered Nautilus mollusk (*Nautilus pompilius* – in Lat.) is a Golden Section spiral.

Let us provide examples of parties' statements about each of the points above.

The term "golden section" is ancient

- **pro**

"This number α[35] is so intriguing a number that it was known to the ancient Greeks at least sixteen centuries before Fibonacci. They called it the *Golden Section*, …." [26; p. 241]

[34] The "golden rectangle" is a rectangle with sides in golden ratio φ.

[35] α — designates the golden coefficient φ in Koshy's book. — *Footnote ours*.

- **contra**

We note at once that φ, the number, could not be known to the ancients since the decimal fraction notation 1.618 did not exist. In his article "Misconceptions about the Golden Ratio" professor George Markowsky writes [36; p. 4]:

> " Many people assume that the names "golden ratio" and "golden section" are very old. ⟨…⟩ However, the use of the adjective "golden" in connection with Φ is a relatively modern one. Even the term "divine proportion" goes back only to the Renaissance. ⟨…⟩
>
> D.H. Fowler [36] [**Fol**; p. 146] gives the following history.
>
> It may surprise some people to find that the name 'golden section,' or more precisely, *goldener Schnitt*, for the division of a line AB at a point C such that $AB \cdot CB = AC^2$, seems to appear in print for the first time in 1835 in the book *Die reine Elementar-Mathematik* by Martin Ohm, the younger brother of the physicist Georg Simon Ohm."

The Great Pyramid of Giza and φ

- **pro**

> "Before the Greeks, the ancient Egyptians used it (Golden Section — *the remark is ours*) in the construction of their great pyramids. The *Papyrus of Ahmes*, written hundreds of years before ancient Greek civilization existed and now kept in the British Museum, contains a detailed account of how the number was used in the building of the Great Pyramid of Giza around 3070 B.C. Ahmes refers to this number as a "sacred ratio." " [26; p. 241]

- **contra**

> "It does not appear that the Egyptians even knew of the existence of Φ much less incorporated it in their buildings." [36; p. 8]

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[36] Fowler, D. H. (David Herbert Fowler; 1937 − 2004). *A generalization of the golden section*. — Pp.146−158. // The Fibonacci Quarterly, Vol. 20, No 2, 1982. — *Footnote ours*.

We can add the following concerning the Papyrus of Ahmes[37]: in the papyrus among the problems about arithmetic and geometry there are four about pyramids (problems 56−59). There are no detailed accounts of how φ was used in the construction of the Great Pyramid. Nor are there words "sacred ratio." On the Internet one can find this papyrus in a decrypted form, if one enters, for example, "Rhind" (after the name of the first owner of the papyrus) or "Ahmes" as the search terms, and see what it contains.

The Parthenon temple and φ

- **pro**

  "The *Parthenon*, the magnificent building erected by the ancient Athenians in honor of Athena Parthenos, the patron goddess of Athens, stands on the Acropolis. It is a monument to the ancients' worship of the golden rectangle … The whole shape fits nicely into a golden rectangle. Even the reconstruction of the original Parthenon in Nashville, Tennessee, vividly illustrates the aesthetic power of the golden rectangle …"
  [26; pp. 277−278]

- **contra**

  "Measurements of parts of a building, or work of art, have such fuzzy boundaries that it is easy to find phi when ratios close to phi fit just as well. Markowsky demolishes the notion that phi is involved in the proportions of the Great Pyramid of Egypt or in those of the Greek Parthenon. There is not the slightest evidence that the Egyptians, Greeks, or any other ancient people, used phi in any of their buildings or art."
  [18; pp. 244−245]

Mario Livio on pages 73−74 of his book "The Golden Ratio: The Story of Phi, the World's Most Astonishing Number" mentions

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[37] The papyrus represents a collection of problems on arithmetic and geometry with solutions. The scribe Ahmes states that he copied it from an earlier document dating from the Dynasty XII of Egyptian rulers, that is, the original dates to the second half of the 1900 BC.

architectural theorist Miloutine Borissavlievitch in connection to the Parthenon temple [38]:

> “Other authors, such as Miloutine Borissavlievitch in *The Golden Number and the Scientific Aesthetics of Architecture (1958)*, while not denying the presence of ϕ in the Parthenon’s design, suggest that the temple owes its harmony and beauty more to the regular rhythm introduced by the repetition of the same column …”

The most aesthetically pleasing rectangle

It would seem a very simple question, that does not require for its verification special equipment or special conditions, but to this time it is not clear with which ratio of the sides a rectangle is the most pleasing to the eyes and if such exists at all.

- **pro**

> “In *Der goldene Schnitt* (1884), Adolf Zeising’s 457-page classic work on the Golden Section, Zeising argued that ″the golden ratio is the most artistically pleasing of all proportions and the key to the understanding of all morphology (including human anatomy), art, architecture, and even music.″
>
> ⟨…⟩
>
> German psychologists Gustav Theodor Fechner (1801−1887) and Wilhelm Max Wundt (1832−1920) provide ample empirical support to Zeising’s claims. They measured thousands of windows, picture frames, playing cards, books, mirrors, and other rectangular objects, and even checked the points where graveyard crosses were divided. They concluded that most people unconsciously select rectangular shapes in the Golden Ratio when selecting such objects. And, of course, such pleasing proportions were the basis of most ancient Greek art and architecture.” [26; pp. 273−274]

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[38] Livio, Mario. *The Golden Ratio: The Story of Phi, the World’s Most Astonishing Number*. New York : Broadway Books, 2003. — 294 p.

- **contra**

“The most persistent misconception is the belief that the "golden rectangle," a rectangle with sides in golden ratio, is the most aesthetically pleasing of all rectangles. The first effort to prove this was undertaken by Gustav Fechner (1801−1887), …” [18; p. 244]

“In the experiments I have conducted so far, the most commonly selected rectangle is one with a ratio of 1.83 …” [36; p. 14]

Artists and *φ*

- **pro**

“Since the golden rectangle is the most pleasing rectangle, countless artists have used golden rectangles and their magnificent properties in their work. ⟨…⟩ Leonardo da Vinci (1452−1519) painted *St. Jerome* to fit very nicely into a golden rectangle; art historians believe that da Vinci deliberately painted the figure according to the classical proportions he inherited from the Greeks.” [26; pp. 275−276]

- **contra**

“Because Leonardo da Vinci illustrated one of the earliest books on phi, *De Divina Proportione*, by Luca Pacioli, phi cultists have imagined that the ratio was intentionally used by Leonardo in many of his paintings. In every case the application of golden ratios to a Leonardo painting is extremely arbitrary and obtained only by fudging. Parts of a figure will extend beyond the borders of the imagined rectangle, and other parts will fail to touch the borders. There is no evidence that Leonardo da Vinci, or any other Renaissance artist or sculptor, used phi in his work. This does not apply to the twentieth century. A few modern architects, painters, and even music composers, fascinated by the golden ratio, have made deliberate use of it.” [18; p. 245]

Human body and φ

- **pro**

"Studies have shown that several proportions of the human body exemplify the Golden Ratio. … AE/CE = Height/Navel height ≈ α and CE/AC = Navel height/Distance from the navel to the top of the head ≈ α." [26; pp. 249−250]

- **contra**

"Some authors claim that the human body is designed according to the golden ratio. ⟨…⟩ While it might be entertaining to compute the ratio of many people's heights to the elevations of their navels, I did not spend much time on this effort. I did compute the ratios for the four members of my immediate family: 1.59, 1.63, 1.65 and 1.66." [34; p. 15]
"Phi buffs are also fond of asserting that on most men and women the navel divides their height in a golden ratio. Indeed, this was one of Pacioli's claims. Why nature would arrange this is never made clear. In any case, it isn't true. Navel height, in relation to body height, varies considerably with race and locale, covering a range that of course includes 1.618." [18; pp. 245−246]

Nautilus pompilius

- **pro**

The assertion that the spiral shell of the *Nautilus pompilius* is a golden section spiral or the "golden" spiral is encountered so often, that it can be attributed to folklore. It is asserted without any references to sources and required explanations, so that the reader is left completely puzzled as to what to measure in the shell and which to divide by which to get 1.618. That is, the information about the shell travels from publication to publication, but it is impossible to locate the source of the information. For example, in the book for elementary and middle school teachers by Buxton and Provenzo on page 21 one can read the following (the references to the sources and required explanations are absent here as well):

> “Did you know that chambered nautiluses and spiral galaxies both conform to the mathematical and architectural ratio known as the Golden Mean?” [39]

- **contra**

> “The nautilus is definitely not in the shape of the golden ratio. Anyone with access to such a shell can see immediately that the ratio is somewhere around 4 to 3. In 1999, I measured shells of *Nautilus pompilius*, the chambered nautilus, in the collection at the California Academy of Sciences in San Francisco. The measurements were taken to the nearest millimeter, which gives them error bars of ±1 mm. The ratios ranged from 1.24 to 1.43, and the average was 1.33, not Φ (which is approximately 1.618).” [14; p. 127]

Fibonacci numbers in nature

The literature contains many cases of the Fibonacci numbers in nature. Let us provide a few.

> “Mature sunflowers display Fibonacci numbers in a unique and remarkable way. The seeds of the flower are tightly packed in two distinct spirals, emanating from the center of the head to the outer edge … One goes clockwise and the other counterclockwise. Studies have shown that although there are exceptions, the number of spirals, by and large, is adjacent Fibonacci numbers; usually, they are 34 and 55. Hoggatt reports a large sunflower with 89 spirals in the clockwise direction and 55 in the opposite direction, and a gigantic flower with 144 spirals clockwise and 89 counterclockwise.” [26; p. 19]

> “The scale patterns on pinecones, artichokes, and pineapples provide excellent examples of Fibonacci numbers. The scales are in fact modified leaves closely packed on short stems, and they form two sets of spirals, called *parastichies*. Some

[39] Buxton, Cory A. and Provenzo, Eugene F. *Teaching science in elementary and middle school: a cognitive and cultural approach*. Los Angeles : SAGE Publications, 2007. — xxviii + 395p.

spirals are clockwise and the rest are counterclockwise, as on a sunflower. Spiral numbers are often adjacent Fibonacci numbers." [26; pp. 20−22]

"The number of petals in many flowers is often a Fibonacci number. ⟨…⟩ Most daisies have 13, 21, or 34 petals; there are even daisies with 55 and 89 petals." [26; p. 17]

The article "*Fibonacci-Tribonacci*" from the journal "*The Fibonacci Quarterly* ," 1963 [16; p. 70] contains a drawing depicting two trees. The number of branches, including the trunk, at various distances from the ground, form for one of the trees a number sequence of the classic Fibonacci series of type (1), while for another — a tribonacci sequence. Interesting, what kind of arrangement of tree branches is most often encountered in the nature and is there even any kind of dominant arrangement at all?

A lot of material on the golden section and Fibonacci numbers is contained in the book *The Golden Ratio* by Mario Livio and in the book *Fibonacci and Lucas Numbers with Applications* by Koshy [26]. To familiarize oneself with the opinions and the arguments of skeptics it is sufficient to read Markowsky's article *Misconceptions about the Golden Ratio* [38] or his book review [40] of *The Golden Ratio …* by Mario Livio, or the article *The Cult of the Golden Ratio* by Martin Gardner [18].

Not for the sake of denunciation

By providing the facts and the opinions of the sides, it is not our goal to accuse anyone of anything. Rather we wish to warn the reader against uncritical acceptance of information on the given themes. In society there exists a division of labor, where each performs their own task. So some people trust in the results of the studies of others, and this is normal. Sometimes, however there is an attribution of reality to what one wishes were true and some wish to see the golden section in everything that is close to the ratio of "one and a half."

[40] Book Review : The Golden Ratio. Reviewed by George Markowsky (The Golden Ratio. Mario Livio). — Pp. 344–347 // Notices of the AMS. Vol. 52, Number 3, 2005.

The proponents are correct in drawing our attention to the facts discovered by them of φ and Fibonacci numbers around us (and in ourselves). However they are not obliged to immediately decide whether the discovery is regular, often encountered, or a random fact, or rare event. The clarification of this can be left to another investigation. True, φ and the classic Fibonacci number sequence find themselves in the surrounding us reality, but are they more numerous than other coefficients and sequences? The data available today is insufficient to draw any kind of conclusions.

So far it is clear that from the mathematical point of view there are no doubts as to the wonderful properties of φ and the Fibonacci numbers, however from the point of view of breadth and preference of their manifestation in the world around us the issue is ambiguous and needs further clarification. The question about the aesthetic dominance of φ remains an open one.

The reader is bewildered

After reading such contradictory opinions about the golden section and Fibonacci numbers presented by us above, the reader will most likely be left confused as to the true state of things in the given area.

Let us try to figure out for ourselves, for example, the situation with the spiral shell of the *Nautilus pompilius* mollusk.

Spiral

If we have a picture of the spiral, then we can assess the degree of its curvature by calculating the ratio k of the radius vectors, differing in their positions by one turn, i.e. by 360°. For example, for the pictured in Fig. 5 spiral F this could be the ratio $k = OE : OG$, where OE is the length of the greater vector of the spiral and the OG of the lesser, and where O is an origin (center) of the spiral. The smaller the k, the closer together are the turns of the spiral. If this coefficient is equal to φ ($k = \varphi$), the spiral is going to be the Fibonacci spiral. However, the values of the coefficients k

calculated by us in this way for thirty *Nautilus pompilius* shells, found in the form of photographs on the Internet, gave an average value of the coefficients k on the order of three. More precisely — just a little over three.

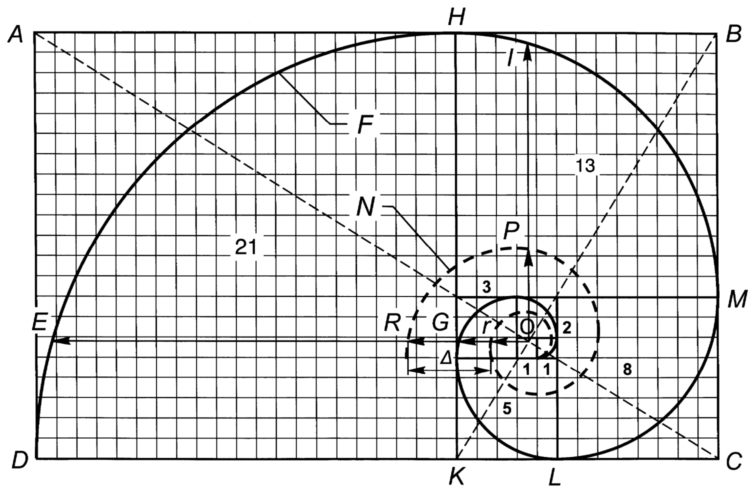

Fig. 5. Fibonacci spiral F and the spiral N of *Nautilus pompilius*

What did the proponents and skeptics measure, if some got $k = \varphi \approx 1.618$, while others, as, for example, professor Clement Earl Falbo, the value $k = 1.33$? [14; p. 127]

We investigate further. Let us consider how, for example, one can construct a Fibonacci spiral.

Fibonacci spiral

Let us join together two squares with the sides of unit length — in Fig. 5 they are squares the size of one grid cell above the numbers 1 and 1. From one of the two points of contact of the squares' vertices draw in one of them (or in both) a quarter circle curve with unit radius as shown in Fig. 5. Adjoin to these squares another square with the sides two units long — in Fig. 5 it is above the unit squares and to the left of number 2 — and draw inside of it a quarter circle curve with the radius of two units as an extension of the existing curve. To these three squares we add a fourth one, with the sides of length 3 units — in Fig. 5 it is on the left of the three squares and under the number 3 — and draw inside of it a quarter circle curve with the radius of three units, and so on. We obtain a spiral *F*, as shown in Fig. 5.

It is common to call this spiral the Fibonacci spiral since it is built on the basis composed of squares, the sides of which form the Fibonacci number sequence of type (2) — in Fig. 5 these squares are 1, 1, 2, 3, 5, 8, 13 and 21 — and the ratio of the sides of the squares, and respectively the ratio of the radii lengths, drawn successively by the quarter arcs of the spiral, tends to the value φ as the sizes of the squares and the lengths of the radii increase. In this case, as we can see, the radius vectors being compared are not those with an angle of 360° between them, as we originally supposed, but 90°. For example, in Fig. 5 it can be vectors *OE* and *OI*. It is precisely for this case, as the skeptics point out, that the average ratio k of the radii (coming from the center of the spiral), for the *Nautilus Pompilius* has the value of about 1.3.

We can draw a Fibonacci spiral using the equation of the spiral in polar coordinates:

$$\rho = ae^{m\theta}. \tag{23}$$

Here ρ — ray or radius vector, emanating from the pole or the center of the spiral;
e — base of the natural logarithm;
θ — counterclockwise angle in radians between the starting and current position of the radius vector;
m and the positive number a — coefficients.

If we let $a = 1$ in (23), we will get a spiral, the initial radius vector of which, as that of the spiral *F* in Fig. 5, will have a unit length when $\theta = 0$.

Assuming that, when rotated by 90° or $\theta = \pi/2 \approx 1.57079$ radians, radius vector ρ increases by $\varphi \approx 1.618$ times; then from

$\rho(90°) = \varphi = e^{m \cdot \pi/2}$ we find that $m = 2/\pi \cdot \ln\varphi = 0.3063489\ldots$ and the equation of the spiral will be the following:

$$\rho = e^{2/\pi \cdot \ln\varphi \cdot \theta} \tag{24}$$

or with precision, for example, to five decimal places,

$$\rho = e^{0.30635\theta}.$$

Since the spiral of type (24) holds throughout its whole length the ratio equal to φ between vectors which are positioned at 90° to each other, it is often called "golden" to distinguish from the Fibonacci spiral, which reaches the ratio only in the limit.

So the "Fibonacci spiral" is a flat spiral, whose radius vector (the distance from the center of the spiral to the point on the spiral) when rotated by 90° changes in magnitude by approximately a factor of φ, and gradually approaches φ as the lengths of the radius vectors increase and reaches φ only in the limit, while the "Fibonacci *golden* spiral" or "spiral of the golden section" or "golden spiral" is a flat spiral, the radius vector of which under the rotation by 90° changes in length by exactly a factor of φ throughout the whole of the spiral.

The origin *O* of the spiral can be found by drawing diagonals in appropriate related by magnitude rectangles, for example, in Fig. 5 these are the dotted intersecting diagonals *AC* and *BK* in the rectangles *ABCD* and *KHBC*. They can also be the lines *DM* and *HL*, not shown in the Fig. 5 so as not to clutter the figure.

It is much harder to draw the spiral using the equation (24), than to draw it using piecewise arc approximation.

The "golden" spiral can be constructed in the same way as the Fibonacci spiral in Fig. 5, if as the initial rectangle *ABCD* we take the "golden" rectangle, that is, rectangle with the ratio of the sides *AB* to *BC* equal to φ. For this in the "golden" rectangle ABCD we allocate the square *AHKD*, while in the formed "golden" rectangle *HBCK* we allocate a square again and so on. In each square we draw a quarter arc as in Fig. 5, and as a result we get a "golden spiral," more precisely a piecewise-arc approximation of the golden spiral. There is no sense in drawing the spiral this way in order to improve the visual perception of it, as only the complexity of the construction increases. So if we start the construction from the golden rectangle with the sides $21 \times 21\varphi = 21 \times 33.978\ 713\ 76\ldots$, whose first square will have the dimensions 21×21, as in Fig. 5, then we will obtain the following series of number values for the sides of the squares:

> $AD = AH = 21$; $HB = BM = AH : \varphi = 21 : \varphi \approx 12.98$, since $AB : AD = AB : AH = AH : HB = \varphi$, as follows from the properties of the golden rectangle and the division of the segment AB in Fig. 4; $MC = LC = 12.98 : \varphi = 21 : \varphi^2 \approx 8.02$; $KL = 21 : \varphi^3 \approx 4.96$; and so on: $21 : \varphi^4 \approx 3.06$; $21 : \varphi^5 \approx 1.89$; $21 : \varphi^6 \approx 1.17$.

As we can see it is essentially the sequence 21, 13, 8, 5, 3, 2 and 1. For those who believe that the variations in the radii of the arcs from the true values in the smaller squares are already visible, we can recommend not drawing the arcs at all in the squares with sides 1×1 and 2×2.

In conclusion, let us add that the spirals, drawn with the three methods above, are visually indistinguishable — the differences in the curves are seen only if all three spirals are superimposed upon each other in one illustration.

What shall we measure?

How did proponents get φ in the mollusk's shell? We found the following indication: in order to get φ, it is necessary to divide the length of the larger of the vectors by the difference between the vectors, differing by one spiral revolution. The meaning? Apparently it is analogous to the division of the segment AB by CB in the Fig. 4. It is sometimes asserted that the ratio of the volumes of adjacent mollusk's chambers is equal to φ, but no one explains how they calculated the volumes of such complex shapes, as that of a chamber.

We calculated the ratio of the greater radius vector, for example, OR in Fig. 5, to the difference $\Delta = OR - Or$ between vectors of the thirty *Nautilus Pompilius* shells mentioned above. Only in one case out of thirty the ratio was equal to φ; the average value of the ratios was equal to 1.52, while the bounds on the changes of the ratios were $k = 1.48 - 1.618$. So even for the measuring method proposed by the proponents, the ratio being equal to φ for *Nautilus Pompilius* is not a law of nature, but a very rare event. Along the way we found out, that the ratio of radius vectors, with 180° between them, in these thirty shells yields the average value of the coefficient $k = 1.76$.

Thus, we have three spirals, one of which we should call Fibonacci spiral: the spiral in which $k = \varphi$ when its vectors differ in their position by one rotation; the spiral with vectors the angle between which is equal to 90°; and the spiral with the ratio of the greater vector to the difference of vectors, when the vectors differ by a turn of the spiral.

The most logical would be to call the *Fibonacci spiral* that, in which $k = \varphi$ when its radius vectors differ in their positions by one turn, but since the name "Fibonacci spiral" is already established for the spiral with the ratio of vectors tending in the limit to φ and an angle between which is equal to 90°, that is the way it should be used. In Fig. 5 this is the spiral *F*.

The *Nautilus pompilius* spiral is shown in Fig. 5 as a dashed curve *N*. The ratio of the radius vectors, rotated by 90° is taken to be 1.33 as in Falbo [14; p. 127], who specially worked on measuring the shells of *Nautilus pompilius*. The equation of this spiral with the precision, for example, of up to two decimal places:

$$\rho_N = e^{0,18\theta}.$$

Both spirals *F* and *N* are constructed in Fig. 5 in the same scale, so that the difference between them can be clearly seen. Both initial radius vectors are equal to one and superimposed. Initial position (direction) of the vectors in Fig. 5 is down from the point *O*. The spirals in the figure are made by the method of piecewise arc approximation.

As can be seen in Fig. 5 the spiral of the shells of *Nautilus pompilius* mollusk is not a Fibonacci spiral. Anyone can, just as we did, perform the required measurements and calculations, having made up one's mind as to what they will divide by what, and find out in this way, if and how often the coefficient φ occurs in the shells of *Nautilus pompilius*.

If we do not use the method of piecewise arc approximation then the initial exact data for the construction of the spirals can be taken from Table 1.

Table 1

Comparative lengths of vectors of the spirals: "golden" and *Nautilus pompilius*

θ	0 (0°)	π/2	π	1½ π	2π (360°)	2½ π	3π	3½ π (1 ¾ trn.)
ρ_G	1	1.618	2.618	4.236	6.854	11.09	17.94	29.03
ρ_N	1	1.33	1.769	2.352	3.129	4.162	5.535	7.361

In Table 1:

θ — angle of rotation of radius vectors in radians, in relation to the initial position $\theta = 0$;
ρ_G — value of the radius vector of the "golden" Fibonacci spiral;
ρ_N — value of the radius vector of the *Nautilus pompilius* spiral;
π — value of the angle θ in radians ($\pi = 3.141592\ldots$).

From the table it can be seen that under rotation by 90° vector ρ_G of the "golden" Fibonacci spiral is lengthened by 1.618 times, while the vector ρ_N of the *Nautilus pompilius* spiral — by 1.33 times and already after 1¾ turns of the spirals, the vector ρ_G (vector OE in Fig. 5) is longer than the vector ρ_N (vector OR in Fig. 5) by approximately four times: $29.03 / 7.361 \approx 3.94$.

To the laid out above we would like to add that everything would have been much easier if the proponents (and the skeptics too) explained, what meaning they assigned to a particular term or concept, what exactly and how they measured, so that we are not left guessing on this subject. In other words, as always everything depends on the eternal problem of definition: tell us what you mean by what you are saying, and how one or another result is obtained, and many of the disputes and controversies will die by themselves.

Stages of research

The discovery of φ and Fibonacci numbers in the world around us is just a first step of the research. Next it is necessary to find out how often the discovered phenomenon manifests itself in comparison to other analogous events. In order to find this out it is necessary to collect and analyze a sufficient volume of statistical material.

For example, in the case of sunflower spirals, it would be better, instead of a general phrase like "the number of spirals in total is adjacent Fibonacci numbers," to provide the data about how many left and right spirals were found in sunflowers on a sufficiently large sunflower field, and perhaps even on a few fields with different varieties of sunflowers. Then process the data and provide it in an appropriate representation, for example, as a table (tables). So the readers will have the opportunity to see for themselves how often a pair of adjacent Fibonacci numbers is encountered among other pairs of numbers of left and right spirals.

The same is true for skeptics: it would be better if instead of an announcement that "the biggest misconception is the belief that the rectangle with sides of the golden ratio is the most esthetically pleasant of all rectangles," they would provide some experimental data, showing the preferences of people in the choice of rectangle with a given ratio of the sides.

The way that, for example, professor Gustav Theodor Fechner (1801−1887) did. Fechner conducted an experiment to find out which shape of a rectangle appeals most to people. Fechner's experiment, or rather the form of the representation of the results of his experiment, can be set as an example both to the proponents of the universality and omnipresence of φ and Fibonacci numbers, as well as to their opponents.

Fechner's experiment with the rectangles

Fechner, assuming, like the above mentioned Adolf Zeising, that the things with the ratio of the sides equal to φ, are the most pleasing to the eye, conducted the following experiment [15; Erster Theil, pp. 193−195].

From a white piece of cardboard ten rectangles were made of the same area — 64 square centimeters each — with the ratios of the sides 1/1 (square), 6/5, 5/4, 4/3, 29/20, 3/2, 34/21 (ratio of the sides equal to φ), 23/13, 2/1 and 5/2.

The white rectangles were laid out on a black surface in an arbitrary orientation and arbitrary order. Interviewees were asked to select the rectangle that was the most pleasing to the eye. There were a total of 228 males (*m*) and 119 females (*w*) interviewed. The results were represented in a table "Tabelle über die Versuche mit 10 Rechtecken" [15; Erster Theil, p. 195]. This table is presented by us in Fig. 6.

Tabelle über die Versuche mit 10 Rechtecken.

(**V** Seitenverhältniss, **Z** Zahl der Vorzugsurtheile, **z** Zahl der Verwerfungsurtheile, **m.** männlich, **w.** weiblich.)

V	Z		z		procent Z	
	m.	w.	m.	w.	m.	w.
1/1 □	6,25	4,0	36,67	31,5	2,74	3,36
6/5	0,5	0,33	28,8	19,5	0,22	0,27
5/4	7,0	0,0	14,5	8,5	3,07	0,00
4/3	4,5	4,0	5,0	1,0	1,97	3,36
29/20	13,33	13,5	2,0	1,0	5,85	11,35
3/2	50,91	20,5	1,0	0,0	22,33	17,22
34/21 ⊙	78,66	42,65	0,0	0,0	34,50	35,83
23/13	49,33	20,21	1,0	1,0	21,64	16,99
2/1	14,25	11,83	3,83	2,25	6,25	9,94
5/2	3,25	2,0	57,21	30,25	1,43	1,68
Summa	228	119	150	95	100,00	100,00

Fig. 6. Preferences when choosing rectangles in the Fechner's experiment

Aside from the results themselves, what is also valuable in this case is the method of presenting the results with an indication of particular distribution of opinions of the participants, or, in other words, how many people chose the rectangle of the given shape.

The method, of course, contains nothing new, but it is exactly what is lacking in the arguments from both sides — the proponents and the opponents of the universality and ubiquity of φ and the Fibonacci numbers.

The data from the Fechner's table provides the reader with a full picture of the distribution of preferences of the experiment's participants, so that the reader may interpret the results of the experiment for themselves and not be a passive recipient of someone else's claim.

From the data of the table, provided in Fig. 6, we have built a graph, presented in Fig. 7.

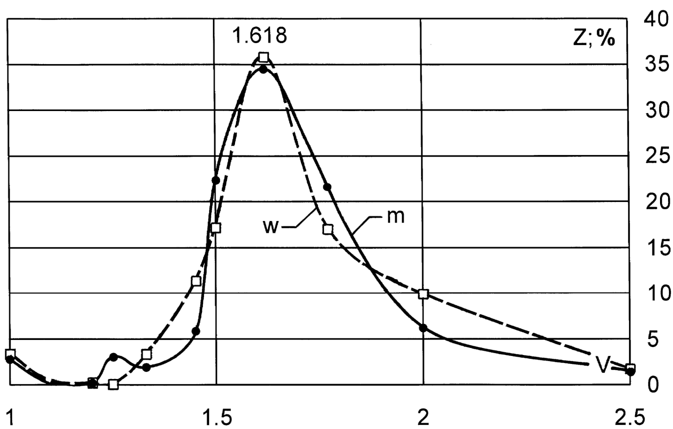

Fig. 7. Preference curves of the choice of rectangles

On the graph, the ratios of the sides of the rectangles (form factor) are laid out along the horizontal axis, and the data from the column "procent Z" (percent of the total number) of the table, provided in Fig. 6, are laid out as dotted markers along the vertical axis.

The markers are connected by the smooth curves: the solid curve *m* is the curve of the preferences for the males, and the dotted curve *w* — for the females. The curves, as we can see, have certain similarities with a normal probability distribution curve.

The graph shows, that the rectangle with $\varphi = 1.618$ is not the sole, unconditionally aesthetically pleasing to the eye, but only more favorable among the others. At the same time the proponents say that the sum of the coefficients 3/2, 34/21 and 23/13 gives ¾ of all the choices provided in the Fechner's table, while opponents doubt even the possibility of existence of an aesthetically preferable rectangle, noting that all people's tastes are different.

A fact is considered scientifically established if it can be reproduced in another place by other researchers. Judging by the statements made by skeptics and, in particular, the results of experiments by Markowsky, in which the most preferred rectangle turned out to be with ratio of the sides 1.83 [36; p. 14], the results of the given experiment by Fechner are not reproducible. Hence his opponents proclaim, that φ has no advantages in aesthetic terms in relation to other coefficients of the shape of rectangles. For example, one can visit the site of the "International Association of Empirical Aesthetics" to read what is being written about this topic. Even Fechner himself in the second part (*Zweiter Theil*) of his book provides data, which can be considered contradictory to his own experiment with rectangles and its conclusion about the aesthetic preference of φ. It has to do with the following.

Ratios of the sides of the paintings

Fechner calculated the ratio of the paintings' sizes using the data from painting catalogs of the nineteen richest in masterpieces galleries in the world [15; Zweiter Theil, p. 314]. For example, only the genre art paintings measured by him numbered nearly a thousand and a half — 755 vertical format paintings and 702 horizontal. The averaged results of his calculations are presented by him in table "VI. Verhältnissmittel von h/b und b/h" [15; Zweiter Theil, p. 291], which we provide in Fig. 8.

From the table it can be seen that for paintings of the vertical format, that is, when the height h of the painting is bigger than its width b (h > b), the average value of the ratio of the paintings' height to width was in the range from 1.248 for landscapes (Landschaft) to 1.258 for still life (Stillleben), while for paintings of the horizontal format (b > h) the average value of the ratio of width to height was in the range from 1.338 for genre (Genre) canvases to 1.388 for still life (Stillleben). As we can see, the ratio of the paintings' sides does not correspond to φ.

VI. Verhältnissmittel von $\frac{h}{b}$ und $\frac{b}{h}$.

	V.-M. $\frac{h}{b}$ h > b	V.-M. $\frac{b}{h}$ b > h	V.-M. $\frac{b}{h}$ Combin.
Genre.	1,250	1,338	1,021
Landschaft.	1,248	1,380	1,282
Stillleben.	1,258	1,388	0,993

Fig. 8. Results of Fechner's measurements of the paintings

Professor Heinrich Carl Franz Emil Timerding (1873–1945) not being a fanatical fan of the golden section, published a small book [54] entitled "*Der goldene Schnitt*" ("The golden section"). In it, on the fiftieth page he provides Fechner's table VI, without the last column and concludes [54; p. 49]:

> "Wie die Tabelle zeigt, stellen sich anscheinend wenigstens mit einiger Annäherung bestimmte Normalwerte heraus, die bei dem Hochformat ungefähr 1¼ und bei dem Querformat 1⅓ betragen."
>
> (As the table shows, it seems that, there are, at least with a certain approximation, definite average norms, which are approximately 1 ¼ in portrait and 1 ⅓ in landscape orientation.)

Why do painters, that is, people in possession of the most developed aesthetic sense of form perception, choose intuitively for their paintings the ratio of the sides not equal to the golden section, but rather one that is close to 1.3, as can be seen from the Fechner's table "VI. Verhältnissmittel von h/b und b/h," while at the same time in the experiment with the rectangles by the same Fechner the people with less developed sense of form preferred φ to all other coefficients? Maybe Fechner did not indicate some of the specific peculiarities of the conducted by him experiment?

Our experiments

Stage one

We repeated Fechner's experiment with rectangles: a small group of the same people — acquaintances and relatives — for the duration of a year, from time to time were asked to choose the most pleasing rectangle to their eyes. The participants were told nothing about the golden section. The interviews were conducted separately with each participant, so no one knew about the choices of others. It turned out that our friends and family preferred rectangles, the ratios of the sides of which mainly grouped around the value 1⅓, which in general correlated with the results of choices by painters for the ratios of sides for their paintings.

Why did our subjects not choose the rectangles with $k = \varphi$, as Fechner's did, or with $k = 1.83$, as Markowsky's did? What was the difference between our rectangles and, for example, the rectangles of Markowsky? In size? In that they were cut out, rather than drawn on paper? The experiments were continued.

Stage two

Since the fractional ratios, used by Fechner in his experiment, are inconvenient to interpret, we drew on paper our ten rectangles with the equal area, but with the ratios of the sides equal to: 1:1 (the square), 1:1.1; 1:1.2; 1:1.3; 1:1.4; 1:1.5; 1:1.618; 1:1.7; 1:1.8 and 1:1.9. If there exists a law for choice preferences, it must manifest itself regardless of what the set of rectangles is. Then we copied our rectangles five times, each time changing the scale so that we obtained a total of six series of ten rectangles. The areas S of rectangles in each series were equal to $36\,\text{mm}^2$; $81\,\text{mm}^2$; $196\,\text{mm}^2$; $6.25\,\text{cm}^2$; $25\,\text{cm}^2$ and $64\,\text{cm}^2$. Standard sheets of writing paper of the European standard A4 (210×297 mm; $k = 297/210 \approx 1.4$), North American standard *Letter* ($8.5''\times11''$ or 216×279 mm; $k = 279/216 = 1{,}294 \approx 1.3$) and *Legal* ($8.5''\times14''$ or 216×356 mm; $k = 14/8.5 = 1.647$) were chosen as the largest rectangles. As we can see, the North American format *Legal* is almost a golden rectangle, but for the purity of the experiment we shortened it by 6.5 mm. To get $k = 1.5$ we took another sheet of *Legal* paper and shortened it to the extent necessary.

The experiment was repeated.

On the black background our rectangles and Fechner's rectangles were laid out, all of the equal area 64 cm^2. From the two collections our friends and family chose, as was expected, equal, if possible, or nearly equal in ratio of the sides rectangles. The choices also coincided among the white rectangles laid out on black background, and those drawn on paper, that is, white on white background.

When choosing between drawn rectangles from a prepared series it turned out that with the smaller size of the rectangles, the longer rectangle was more appealing. So if in the series with an area $S = 64$ cm^2 the appealing rectangle was with $k = 1.4$, then in the series with $S = 36$ mm^2 or with $S = 81$ mm^2 the rectangle with $k = 1.5$ or 1.618 was liked more. Why? Or is this a normal nature of all people, or an anomaly of our group only? If such features are inherent in all people, then how can we talk about a rectangle with the ratio of the sides, equal to φ, being the most pleasing to the eye, if the pleasing ratio depends on the size of the rectangle?

In the case with the sheets of writing paper we asked participants to select a sheet of paper, on which it would be the most comfortable to write and a sheet most pleasing to the eye. In both cases the choices — practical (utilitarian) and aesthetic — coincided and no one chose the format with the ratio φ, as the chosen ratios were usually equal either to 1.3 or 1.4, and some said that they would have liked to have chosen something in-between 1.3 and 1.4. Does this show that something, that is convenient to a person, is also beautiful, or that, as in the second case, people actually make choices based on practical considerations, even though it seems to them, that they are choosing based on aesthetics?

The choice

It is necessary to note, that the preferences of the participants could change within certain limits from interview to interview, that is, the same participant could pick one rectangle today, but after some time — another. Thus there were deviations from the mean, usually within the limits of 0.1 and rarely 0.2. Let us also note such an interesting fact: the one of us, who drew and cut out the rectangles, knew, naturally, how the golden rectangle looked and initially always chose it. However after a four month break in experiments he chose the rectangle with $k = 1.5$, apparently forgetting, how the rectangle with ratios of the sides equal to φ looked. After the next break he chose a rectangle with $k = 1.8$, possibly remembering unconsciously that the last time he chose a very "short" rectangle. In the following experiments he repeatedly chose rectangle with $k = 1.5$ and very rarely with $k = 1.618$. In this way, his preferred choices for the duration of this experiment can be interpreted as initially being realized on the basis of a prior preset, that is, unconscious adjustment for selecting a rectangle, similar to golden, and not an adjustment for selecting one pleasing to the eye.

If this is so, then the researcher must take into account the possibility of influence on the choice of the rectangle of such factor as the preparation for a choice (the Hawthorne Effect): if the person knows how a golden rectangle looks, he may, in force of this, prefer it to the rest. The choice of a square by some in Fechner's experiment — the spike along the vertical axis at the value of 1 in the graph in Fig. 7 — the rise of the curves for figure with ratio of the sides 1:1 — can be explained by the fact that the preference was given to the square for the reasons of perfection of the form, and not aesthetics. In this way, there appears, besides the sizes of the rectangles, one more factor, which must be considered when conducting an experiment — the special attuning process (adjustment) to perceive a particular quality of an object.

The perception

It is natural for people to perceive the object not as a whole, but as one or another of its properties, which at present interests them the most. So, the utilitarian approach allows us to evaluate the usefulness of the thing and the possibility of its use for human needs. The assessment of the ideality of an object provides a notion about the perfection of its form, proportions, regardless of its usefulness. One can perceive objects in terms of "like – do not like" (beautiful – ugly), that is, aesthetically. Aesthetic or aesthetically appealing object is an object evoking a positive feeling from perception; it is that which we like.

Ideal (perfect) objects — square, circle, sphere or cube, for example, — must not necessarily be liked and/or perceived as beautiful. In fact, both agree with the given statement — the proponents of φ, who claim that the most pleasant to the eye ratio is 1:1.618, and not the ideal 1:1, and the skeptics, who say that people usually choose the ratios close to 1.3 [54; p. 49], or 1.83 [36; p. 14]. The ideal (ideality) is something that does not contain any flaws. Each point on the surface of the sphere is at the same distance from the center, the surface of the sphere is perfectly round, flawless — in one word: ideal. Will the picture that is filled with ideal spheres and cubes of different sizes, and painted in different colors, evoke aesthetic pleasure? The landscape, to be perceived as beautiful, aesthetically appealing, must have in the painting a certain amount of nonideality, disorganization.

The beautiful is, rather, a fortunate deviation from the ideal.

The perception of beauty is different for everyone and the boundaries of the beautiful are not clearly defined, blurred. For everyone the "fortunate" deviation will be their own.

It seems, that aesthetic perception requires a special mindset, in absence of which either the utilitarian (rational) or ideal is selected. It is interesting to note that the rectangle with $k = 1.1$ was not selected by anyone (is it ugly from both points of view — idealistic and aesthetic?), so it is like a separating barrier between the ideal and the aesthetic. Did Fechner create a mindset (preadjustment) for the beautiful in participants of the experiment or did he receive the most votes for φ with completely no influence over the selection process? We, at least, failed to convince the participants that φ is the most pleasing to the eye ratio of the sides of the rectangle, even though we tried to do this in the third stage of our experiment.

Stage three

In the third stage before the start of the survey, we told the participants that many consider the rectangle with the ratio of the sides 1:1.618 to be the most pleasing to the eye. This rectangle was shown to everyone, and the participants were asked if they could consider it more aesthetic and pleasing to the eye, than the one they previously chose (if rectangle chosen by them before was not with the ratio of the sides 1:1.618), that is, clearly there was an attempt to influence the choice.

Regardless of the efforts undertaken by us, the initial results of our experiments did not change, everyone continued to consider the rectangle chosen by them to be the most pleasing to the eye. Even Immanuel Kant (1724−1804) in the “Critique of the Power of Judgment” (*Kritik der Urteilskraft* , 1790) noted that no logical proofs or explanations could force a person to recognize as beautiful that which he or she does not like.

Participants were also asked to choose a not too square and not too long rectangle among the rectangles with an area of 64 cm^2. They usually chose a rectangle in the range of $k = 1.5-1.618$. This apparently is the meaning of the coefficient φ. To our question of whether they found it more pleasing to the eye than the one that they usually chose (if their early selections were different from the ones at the current moment), they answered that they did not. And the one of us, who cut the rectangles for the experiments and usually picked in the range of $k = 1.5-1.618$, said so directly that he first looks to see that the rectangles was not too square and not too long and then chooses it. Is such a “mechanical” choice aesthetic? After all, if during a journey before us suddenly opens a beautiful view, we just marvel at the new beauty, and do not try to discern in the newly opened the “not too square and not too long.”

So for us, it remained a mystery, how according to Fechner a rectangle with golden ratio of the sides received preference over the others. Either we could not create in the participants of the experiment an aesthetic mindset, or we could not create a mindset for choosing specifically the “golden” rectangle. Or, perhaps, we could not create a mindset of “not too square and not too elongated”?

Aesthetic upbringing

We did not mention one more factor, which it is necessary to take into account — aesthetic upbringing in the family, in school and in society. If such an upbringing does not exist and the sense of beauty is not given to the individual by nature, then we can hardly hope that the participants will choose the golden rectangle, even if it is objectively the most preferable among the others. In other words, people lacking the sense of beauty, must not participate in an experiment of this type — the same way as color-blind people should not participate in an experiment that determines which color people like the most. There remains, however, a question — how do we sort people according to the criteria of aesthetic suitability, that is, how do we pick participants, such that the results of the experiment conducted with them can be considered correct.

The feeling of the beautiful has to be taught, the makings of the perception of the beautiful have to be cultivated. The process itself is long, continuing for the entire course of human history, and depends wholly upon the society's level of development, and in particular — on those conditions, on that spiritual environment, which it develops in each particular family. So it is completely possible, that the aesthetes of the future will prefer (or, perhaps, not) the golden rectangle to the others.

The reason, we repeated Fechner's experiment, is that we were simply interested in finding out what kind of rectangles are liked by our friends and family. When we saw that our results differed from those of Fechner and Markowsky, we decided to figure out why it happened. However during this we found so many nuances, that we understood that the finding out which rectangle is the most appealing to the eye is not such a simple problem, as we thought initially, and we abandoned this idea.

Our advice

From the provided examples it can be seen, that φ and the Fibonacci numbers are present in the world around us. But it is still unclear whether the discovered facts are of frequent occurrence (regular phenomena), as for example, a strictly defined number of petals in certain flowers or they are accidental (random phenomena), as for example, number φ discovered by us accidentally in a series of measurements of proportions of the *Nautilus Pompilius* mollusk's shell — even that was by using a somewhat peculiar method of measurement, proposed by the proponents. Do the φ and the Fibonacci numbers belong to the set of fundamental principles of the universe, as the proponents state it, or not? Based on provided by us directly opposed views of the parties, it is difficult to make up one's mind with respect to the above facts.

However, each among their own family and friends can conduct the same Fechner's experiment or calculate the ratios of the sides of the houses, walls, ceilings, doors, windows, cabinets, shelves, paintings, suitcases, beds, refrigerators, televisions, sheets of writing paper and other rectangular household items and find this way, how often the coefficient φ is encountered in our everyday life.

One can photograph the trees and count the number of branches, to find out if the Fibonacci series is realized for them.

The normal process of studying

After the Renaissance interest in the golden section waned and for more than two hundred years this ratio was left to oblivion and only in the second half of the 19th century there began to appear publications on finding its new properties and the discovery of it in certain events and objects.

In general, in relation to the golden section and Fibonacci numbers, we can say: there is a normal process of studying the phenomenon and there is a criticism noting the arising extremes and mistakes. As usual, the process of understanding something new develops in the direction from the appearance of the first delights regarding the discovery to the determination of the real value of the discovery and placing it into the appropriate for its value niche of knowledge.

Chapter VI

ABOUT FIBONACCI AND HIS WORKS

When one is writing about Fibonacci, they say something in the spirit of the following:

> "Fibonacci was the greatest Christian mathematician of the Middle Ages, and the mathematical renaissance in the West may be dated from him."

This is how a distinguished historian of science George Alfred Leon Sarton (1884−1956) writes about Fibonacci [43; p. 611]. Another great historian of mathematics of 19th century Moritz Benedikt Cantor (1829−1920) expressed himself in the following way [11; p. 53]:

> "Leonardo war ein gewandter Rechner, ein feiner Geometer, ein geistreicher Algebraiker, wie es vor ihm nur Vereinzelte gab; er wusste die Algebra auf geometrische Fragen anzuwenden, wie kaum Abû'l Dschûd (Bd. I. S. 715) es verstand; er war endlich ein geradezu schöpferischer Zahlentheoretiker.
>
> Ein glänzendes Meteor taucht er auf, wie ein Meteor verschwindet er!"

> (Leonardo was a skilled arithmetician, a fine geometer, an ingenious algebraist, as there were only few before him; he knew how to apply algebra to geometrical problems as hardly Abu'l Djud (Bd. I. S. 715) was able; finally he was an absolutely creative number theoretician.
> As the brilliant meteor he appears, as a meteor he disappears!)

These two pieces of evidence from the two authoritative figures are enough to understand who Fibonacci was.

Let us provide a few statements of some other authors about the works of Fibonacci.

Liber Abaci

Rufus Buel McClenon, professor (1852−1920):
"Thus the result of Leonardo's travels was the monumental *Liber Abaci* (1202), the greatest arithmetic of the middle ages, and the first one to show by examples from every field the great superiority of the Hindu-Arabic numeral system over the Roman system exemplified by Boethius. It is true that Leonardo's *Liber Abaci* was not the first book written in Italy in which the Hindu-Arabic numerals were used and explained, but no work had been previously produced which in either the extent or the value of its contents could for a moment be compared with this." [39; p. 2]

Nikolai Nikolaevich Vorobiev [55; p. 1]:
"*Liber abaci* is a voluminous compendium including almost all the arithmetical and algebraic knowledge of those times. The book played an important part in the development of mathematics in Western Europe through many subsequent centuries. In particular, it was from this book that Europeans became acquainted with the Hindu-Arabic numerals."

John Derbyshire (pronounced "John DAH-bi-shuh"; journalist and a writer with a mathematical background):
"*Liber abbaci* was, by the standards of its time, wonderfully innovative and very influential. For 300 years it was the best math textbook available that had been written since the end of the ancient world. It is often credited with having introduced "Arabic" (that is, Indian) numerals, including zero, to the West." [*Unknown Quantity* [41]; p. 68]

[41] Derbyshire, John. Unknown Quantity : A Real and Imaginary History of Algebra. — Washington, D.C. : Joseph Henry Press, 2006. — viii + 374 p. + [8 p.]

Practica Geometriae

Rufus Buel McClenon:
"This contains a wide variety of interesting theorems, and while it shows no such originality as to enable us to rank Leonardo among the great geometers of history, it is excellently written, and the rigor and elegance of the proofs are deserving of high praise." [39; p. 2]

Liber Quadratorum

Olry Terquem (the year of 1856):
"Nous avons déjà vu que c'est une question proposée par Jean de Palerme qui a engagé Fibonacci à composer le Traité des Carrés dédié à l'empereur Frédéric II. C'est le monument arithmologique le plus précieux que nous ait transmis le moyen-âge, et où l'auteur, successeur de Diophante et des Arabes, se montre esprit indépendant, original, créateur et digne précurseur de Fermat, ou plutôt du XIII^e siècle il faut descendre jusqu' au XVII^e pour rencontrer dans Fermat un second Fibonacci." [53; p. 136]

(We have already seen that it was a question proposed by John of Palermo which urged Fibonacci to compose the *Treaty of Squares*, dedicated to the Emperor Frederick II. This is the most invaluable monument of science about numbers which passed down to us from Middle Ages, where the author, the successor of Diophantus and Arabs and a worthy precursor of Fermat, shows independent, original and creative mind, or rather from thirteenth century we have to move to the seventeenth to meet in Fermat a second Fibonacci.)

McClenon (the year of 1919):
"In the *Liber Quadratorum*, Leonardo has given us a well-arranged, brilliantly-written collection of theorems from indeterminate analysis involving equations of the second degree. ⟨…⟩ At all events, considering both the originality and power of his methods, and the importance of his results,

we are abundantly justified in ranking Leonardo of Pisa as the greatest genius in the field of number theory who appeared between the time of Diophantus and that of Fermat." [39; p. 3, p. 8]

Thomas Koshy (the year of 2001):
"… *Liber Quadratorum* earned Fibonacci his reputation as a major number theorist, ranked between the Greek mathematician Diophantus (ca. 250 A.D.) and the French mathematician Pierre de Fermat (1601−1665)." [26; p. 3].

Laurence Sigler (the year of 2002):
"It is his *Liber quadratorum*, or *The Book of Squares* [Si], that offers best testimony to his power as a mathematician. This work can be said to stand between the work of Diophantus and the work of Pierre Fermat in the theory of numbers." [44; p. 5 (Introduction)]

Comments are superfluous.

Flos, Epistola, Liber Quadratorum

"The other works of Leonardo of Pisa that are known are *Flos*, a *Letter to Magister Theodorus*, and the *Liber Quadratorum*. These three works are so original and instructive, and show so well the remarkable genius of this brilliant mathematician of the thirteenth century, that it is highly desirable that they be made available in English translation." [39; p. 2]

Aspects that are touched upon

In assessments of the significance of Fibonacci and his works, one way or the other usually the following aspects are touched upon:

- Fibonacci's qualification;
- Arabic numerals and Fibonacci;
- priority of Fibonacci in matter of Fibonacci numbers;
- Fibonacci's contribution and the degree of influence of his works.

Fibonacci's qualification

There are no doubts that Fibonacci was a mathematician of the highest level. First of all, there are no reasons not to trust in above statements of authorities, and second, Fibonacci's works speak for themselves.

Who was Fibonacci, as an expert of the highest level: ingenious problem solver (calculator); researcher who discovered new knowledge; a knowledgeable compiler, that is, erudite mathematician, who collected into one and presented in writing the mathematical knowledge known at that time, or a little bit of everything? It seems that Fibonacci possessed all of these qualities.

Popular belief

A widely spread opinion is that it is hard to overestimate the value of Fibonacci's books for the development of mathematics and the dissemination of mathematical knowledge in Europe, that by these books, superior to the level of Arab and medieval European writings, the mathematics was taught almost to the time of Descartes (17th century) and Euler (18th century); that Fibonacci was the first, to introduce the Hindu-Arabic number system into Europe, eliminated the use of complex Roman numerals and thereby made math more accessible to the public; that after the writing of his works new schools opened in Italy, to teach counting by the new system and that for these schools the textbooks were written based on the works of Fibonacci.

"Arabic" numbers and Fibonacci

When it comes to the role of Fibonacci in introduction of Arabic numerals in Europe, some say that it was Fibonacci who introduced the Europeans to the Indo-Arabic positional decimal system of counting and embedded it in Europe, others — that Fibonacci was one of the first who did it. Sigler, for example, writes [44; p. 4]:

> "It was Leonardo's purpose to replace Roman numerals with the Hindu numerals not only among scientists, but in com-

> merce and among the common people. He achieved this goal perhaps more than he ever dreamed."

At the beginning of 13th century, Arabic numerals and calculations using numbers with positional values, were not at all unusual for the scholars of the West. According to the prevailing point of view, these numbers came to Europe from India. As Sigler notes: [44; p. 3]:

> "Knowledge of the Hindu numerals began to reach Europe in the second half of the tenth century through the Arabs by way of Spain, however their usage was still not a general practice at Leonardo's time."

According to an alternative point of view — Nicholas Bubnov (Bubnov Nikolaj Mikhajlovich; 1858−1943), George Rusby Kaye (1866−1929), Bernard Carra de Vaux (1867 −1953) — "Arabic" numerals and the decimal positional number system were not borrowed from the Arabs, but were known to classical antiquity — were, so to speak, "local" in origin and known long before Fibonacci [28; p. 182, p. 192].

Alfred Lieber on the Arabic numerals, writes:

> "Leonardo was not, of course, the first European to draw attention to the Indian numerals. Gerbert, the later Pope Sylvester II, had already described them at the end of the tenth century, albeit without the zero, and in the first half of the twelfth century al Khwarezmi's treatise on arithmetic was translated by either Adelard of Bath or Robert of Chester, under the name *Algoritmi de Numero Indorum*." [29; p. 243]

The oldest definitely dated European document known to contain the Hindu-Arabic numerals is a Latin manuscript the "Codex Conciliorum Albeldensis seu Vigilanus " or "Codex Vigilanus," or "Codex Albeldensis," or "Albelda Codex," written in 976 A.D. in the Albelda Cloister not far from Logroño in northern Spain. [50; pp. 137−138], [56; p. 21].

The transition from the additive Roman numerals to the positional Hindu-Arabic numeral system came slowly over a period of centuries. By 1375 the Hindu-Arabic numerals had a firm hold on Europe. A number of interesting documents show the change actually taking place. So a set of family records in the British Museum shows the first child being born in Mijc.Lviii, that is, in 1258 (the ijc being a form of writing 200), the second child was born in Mijc.Lxi, the third in Mijc.63, while the fourth and fifth children were born in 1264 and 1266. [56; pp. 25−26]

Either way, it appears that Fibonacci did not embed Hindu-Arabic numerals in Europe — this would have been a very strong claim for any one person: these numbers were already known in Europe before him and were coming into use gradually, over a period of a long time. What remains is to find out, how Fibonacci, or rather, his works have contributed to their proliferation.

Not the only book

Liber abaci was not the first and not the only book written in Europe to describe the new numeral system. Other instructors and mathematicians also wrote books of *abaco* for use in the school. These books vary from primitive rule manuals up to mathematics books of quality, but none was so comprehensive, theoretical, and excellent as the *Liber abaci* of Leonardo Pisano [44; p. 5]. In the times of Fibonacci, there existed, for example, translations from Arabic of books on algebra and arithmetic by the Arab scholar al-Khwarizmi, manuscripts by Alexander de Villa Dei and John Sacrobosco.

Arithmetic of Al-Khwarizmi

The first work that presented Europe with an Hindu-Arabic decimal positional number system, is a manuscript on arithmetic by the great Arab scholar Muhammad ibn Musa al-Khwarizmi. The work was written in about 825 and translated into Latin in about 1120 [56; p. 24] (in 1130, or in 1140 — according to other sources). The Arabic original of the manuscript is lost and the original Arabic name of the manuscript is unknown.

The Latin manuscript is untitled and is commonly referred to by the first two words, with which it starts, as "Dixit algorizmi" ("So said Al-Khwarizmi") or as "Algoritmi de numero Indorum"

(*The Hindu Art of Reckoning*), the name given to the manuscript by Baldassarre Boncompagni in his work "Trattati d'aritmetica (I. Algoritmi de numero indorum)" published in 1857.

From the translator's Latin interpretation of the Arabic name Al-Khwarizmi as "algorizmi" (Dixit algorizmi = said Al-Khwarizmi) originated the term "algorithm," and for a long time the writings on the art of counting with the use of the Hindu-Arabic positional decimal number system were also called algorisms (*algorism*; *algorismi*) or algorithms (*algoritm*; *algoritmi*).

The full name of Al-Khwarizmi (Al-Khawarizmi, or Al-Khowarizmi, or Al-Khorezmi) is Abu Abdullah Muhammad ibn Musa al-Khwarizmi al-Majusi or Abu Jafar Muhammad ibn Musa al-Khwarizmi al-Majusi, which means "Muhammad, father of Abdullah (or father of Jafar), son of Musa, from Khwarezm, of the magicians"; the presumed dates of birth and death are 780 and 850. Khwarezm (Chorezm, etc.) was an ancient state in Central Asia.

It is often pointed out that the arithmetic of Al-Khwarizmi was responsible for the diffusion of the Indian system of numeration in the Middle-East and Europe. Though, it may be not proper to assign to one book or one person such responsibility: in those distant times there was a brisk trade between European states and the East, but the trade routes are also the paths of the knowledge transfer; therefore, it would be more correct to say that the number of pathways of penetration of the knowledge about Indian system of counting, same as of the carriers of the knowledge, was sufficiently large.

Smith and Karpinski note [50; p. 99]:

> "From what has been said of the trade relations between the East and the West, and of the probability that it was the trader rather than the scholar who carried these numerals from their original habitat to various commercial centers, it is evident that we shall never know when they first made their inconspicuous entrance into Europe."

One of such carriers of knowledge can be considered Fibonacci himself, who, being with his father in Algeria and then in different countries, first acquired his knowledge and then "delivered" it into Italy and put it in writing there.

Poem by Alexander de Villa Dei

Around 1220 (or around 1225) a well-known French medieval grammarian and mathematician Alexander de Villa Dei wrote in Latin the treatise "Carmen de Algorismo" [20; pp. 73−83]. The formal translation of the title of the work is "The Poem of algorism" meaning "The poem on the Hindu-Arabic decimal positional number system." The more acceptable and convenient title is "The Poem on the Art of Reckoning." The manuscript treats of the fundamental operations with integers. As with other math textbooks at that time, the writing is no more than a collection of rules without proofs and without numerical examples.

> "The *Carmen de Algorismo* of Alexander de Villa Dei was written in verse, as indeed were many other textbooks of that time. That it was widely used is evidenced by the large number of manuscripts extant in European libraries."[50; p. 134]

Alexander of Villedieu-les-Poêles (Alexander de Villa Dei — in Latin) was born, presumably, in 1160 or in 1170 in Villedieu-les-Poêles and died around 1240 or around 1250. Villedieu-les-Poêles (pronunciation: vildjølepwɑl) is a town and commune in the Manche département, Normandy, northwestern France.

Sacrobosco's "Algorismus"

Around the year 1225 or the year 1250 English astronomer and mathematician Johannes de Sacrobosco or John of Holywood (1195−1256) wrote a small treatise on arithmetic "Algorismus" (*The Art of Reckoning*). Sacrobosco is the Italian translation of the English name Holywood. Many say that Holywood (now Halifax), in Yorkshire, England was the birthplace of Sacrobosco.

Algorismus describes basic operations with whole numbers without any proofs and examples. Other titles of the work: *Algorismus de integris*, *Algorismus vulgaris*, *Opusculum de praxi numerorum quod Algorismum vocant*, *Algorismus domini Joannis de Sacro Bosco*. In the print edition — in the collection "Rara Mathematica," the compiler and editor of which was James Orchard Halliwell-Phillipps (1820−1889), the work by Sacrobosco is presented under the title "Johannis de Sacro-Bosco Tractatus de Arte Numerandi" [20; pp. 1−26], while in the preface to the collection "Rara Mathematica" Halliwel mentions this work as "Johannes de Sacro-Bosco de Arithmetica" [20; p. v].

The work enjoyed a wide popularity as a textbook for universities and continued to be used as such even after the invention of printing. Printed editions are known from as late as the fifteenth and sixteenth centuries. [56; pp. 23−24]

On the influence of Liber Abaci

What role did, among others, the works of Fibonacci play in spreading the Hindu-Arabic positional number system in Europe? To understand this we turn to the work of Suzan Rose Benedict (1873−1942), the first woman to receive a PhD in mathematics from the University of Michigan, USA. Her PhD dissertation was titled "A comparative study of the early treatises introducing into Europe the Hindu art of reckoning." Suzan's aim was precisely to figure out, which works and to what extent contributed to the spread of Hindu-Arabic positional number system in Europe, and how they influenced the development of computational methods [2; p. 120]:

> "It is my purpose therefore, in closing this paper, to discuss briefly some of those algorisms which seemed to me influential in the development of methods of calculation, and to compare their relative importance."

Suzan Benedict examined about forty medieval manuscripts. Here is what she writes:

> "The *Carmen de algorismo* written by Alexander de Villa Dei, was one of the most influential treatises of the period, as is shown not only by the great number of translations and copies extant, but by the similarity of treatment found in many later works. ⟨…⟩ Translations into English, French and Icelandic are known, and commentaries in English and Latin, as well as many Latin copies are to be found in the libraries of Europe." [2; pp. 122−123]

> "The *Alogrismus vulgaris* of John of Sacrobosco is the work of a scholar and a teacher, and well deserves the great popularity it attained. ⟨…⟩ The *Algorismus vulgaris* was not written as a theoretical text for philosophers, but as a practical exposition of the art of reckoning, to be used in the universities of the period. ⟨…⟩ that its value was appreciated is shown by the great number of manuscripts that are to be found in most collections of mathematical works, and by the extensive use of it which was made by later writers."
> [2; p. 123]

Suzan Benedict makes the following conclusion [2; p. 126]:

> "… up to the time of printing the *Carmen de Algorismo* of Alexander de Villa Dei, and the *Algorismus vulgaris* of Sacrobosco were the most widely read of all the Latin works."

Suzan Benedict's conclusion about the work of Fibonacci *Liber Abaci* [2; pp. 119–120]:

> "Though this treatise, from a mathematical point of view, was far superior to the translations of the work of Al-Khowarizmi, it seems not to have exerted so great an influence. This may have been because the western world was not yet ready for so advanced a treatise, or because the monks, copying from monastery to monastery spread the other type of algorism."

Professor Michael Roy Williams echoes Suzan Benedict:

> "The two main works which spread the knowledge of Hindu-Arabic arithmetic through Europe were the *Carmen de Algorismo* (The Poem of Algorism) by Alexander De Villa Dei from about 1220 and the *Algorismus Vulgaris* (Common Algorism) by John of Halifax, better known as Sacrobosco, from about 1250 A.D. Both of these works were based, at least in part, on the works of Al Khowarizmi or one of his successors. They were both designed for use in the European universities then starting up in places such as Paris and Oxford and were not meant to be complete explanations of the sys-

tem; rather they simply gave the basics so that a lecturer could explain them, line by line, to his students. The *Carmen de Algorismo* ⟨…⟩, being only 284 lines[42] long, … was easily copied by scribes, and a hundred copies could be made and distributed in the time it would take to make one copy of *Liber Abaci*. The same was true of the *Algorismus Vulgaris* which was only about 4,000 words long.[43]" [56; pp. 22–23]

> "The *Liber Abaci* was not as influential as it might have been because it was rather large, and thus difficult to copy in the days before printing. It also contained advanced material suitable only for scholars so that it was known only to a few people, none of whom seemed to have had much influence on the methods of calculation used in everyday transactions.
>
> Although the efforts of Fibonacci were of little success, the idea of the Hindu-Arabic numerals gradually spread into Europe. The main source of information was the various translations, or partial translations, of Al-Khowarizmi's *Arithmetic*."[56; p.22]

As we can see, *Liber Abaci* did not attain any wide usage and did not exert any notable influence.

Borrowings

Did Fibonacci borrow from anyone during the writing of his works and did anyone borrow from Fibonacci's works? Of course. Previously accumulated knowledge is always used during the writing of works. About Fibonacci's borrowings Hoyrup, for instance, writes [22; p. 36]:

> "He certainly took his inspiration from many sources, some of which can be identified — as we have seen, the notation

[42] In "Rara Mathematica; … ," London, 1839, referred to by Benedict in her thesis [2; p.12], "Carmen de Algorismo" consists of 285 lines, taking into account the third line of only digits. — *Footnote ours*.
[43] More precisely, there are about 4600 words in issue [20], not counting the title and footnotes, but counting each separate number as a word. — *Footnote ours*.

> for ascending continued fractions emulates that of the Maghreb mathematical school, the beginning of the algebra of the *Liber abbaci* copies creatively but unmistakeably from Gherardo of Cremona's translations of al-Khwārizmī's *Algebra* [Miura 1981], the *Pratica geometrie* from the same translator's version of Abū Bakr's *Liber mensurationum*. Most of his sources, however, are unidentified."

Borrowings from the works of Fibonacci

To find out how often the material from the works of Fibonacci has been used in the later mathematical writings and books for the schools *abbaco*, we can use the results of research studies by Hoyrup. Hoyrup meticulously studied many of the manuscripts that were preserved and survived to our time, and which appeared later than Fibonacci's work *Liber Abaci*. As Hoyrup notes himself [22; p. 39]:

> "Many Italian abbacus treatises, and *all* Ibero-Provençal writings I have had the opportunity to examine, …"

For example, in his article "Leonardo Fibonacci and *Abbaco* Culture. A Proposal to Invert the Roles" [22] Hoyrup analyzes the oldest extant after *Liber Abaci* manuscript — *Livero de l'abbecho* "*secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa*" (*Abbacus book* according to the opinion of master Leonardo Fibonacci). The work appears to have been written in about 1288−1290 in Umbria [22; p. 27]. The given article by Hoyrup was partly included in his mentioned earlier by us book "Jacopo da Firenze's Tractatus *Algorismi* and Early Italian Abbacus Culture." In this book Hoyrup analyses another of the earliest, known after *Liber Abaci* manuscripts, — the manuscript "Tractatus algorismi," written in Montpellier in September 1307 by Jacopo da Firenze (Master Jacob of Florence — in English).

The extensive research conducted by Hoyrup allowed him to draw the conclusion that the borrowings from Fibonacci's work *Liber Abaci*, and also from his other works, were insignificant. On the other hand, the fact of the borrowings itself says that Fibonacci's works or at least some of them were, at one time or another, known to a certain, albeit very limited, number of people.

So, for example, in the very beginning of his treatise on mathematics "Suma de Arithmetica Geometria Proportioni et Proportionalita" (*The Totality of Arithmetic, Geometry, Proportions and Proportionality*), in the section without numbering "Summario de la prima parte principale," the monk and mathematician Luca Pacioli (1445−1517) among the authors, the works of which he used in writing his "Suma …" mentions Fibonacci:

> "E queste cose tutte con lesequenti. siranno secondo li antichi. E ancora moderni. mathematici. Maxime del perspicacissimo phylosopho megarense. Euclide E del seuerin Boetio. e de nostri moderni Leonardo. pisano. Giordano. Biagio da parma. Giouan sacrobusco. e Prodocimo padoano. da iquali in magior parte cauo elpresente volume."

The work *Suma* was printed typographically in 1494. The second edition was released in 1523 and began with the words: "Summa de Arithmetica geometria. Proportioni: et proportionalita: Nouamente impressa In Toscolano su la riua dil …" Since the second edition, as usual in that time, had a very long name, the work today is referred to as "Summa de arithmetica, geometria, proportioni et prportionalita" or simply "Summa."

In the first part of the book in the section "Distinctio prima, Tractatus quartus," on the back of the 13th sheet — the front side of which is in error numbered 15, and not 13, as it should be, — Pacioli mentions Fibonacci and his *Liber Quadratorum* under the name "quadratis numeris":

> "E simili a questa. Le quali domande sonno difficillissime quanto ala dimostratione dela pratica: commo sa chiben la scrutinato. Maxie Leonardo pisano in vn particulare tractato che fa de quadratis numeris intitulato."

Both quotations were borrowed by us from the work *Suma* of the first edition for the year 1494 and provided with the original spelling.

Boncompagni in the footnote numbered (4) on pages 81−84 of his work *Intorno ad alcune opere di Leonardo Pisano*, published in book form in 1854 [6], lists the places where Fibonacci is mentioned in the Pacioli's *Summa*.

As Fibonacci's *Liber Abaci*, *Summa* is a capital work. Pacioli's book *Summa* is mentioned by many algebraists of the 16th century, who, if not from other sources, at least from this one, could find out about Fibonacci. However, as already after a hundred years Kepler knew nothing of Fibonacci, there was clearly a lack of sources of information about Fibonacci.

Translations and copies of Fibonacci's works

We refer again to Hoyrup, who writes [22; p. 35]:

> "A couple of translations from the *Liber abbaci*, one of chapters 14–15, another of most of chapter 12 and a little of chapter 13, go back to c. 1350; another translation of chapters 14–15 can be dated c. 1400, as can a translation of the *Liber quadratorum*; a translation of the *Practica geometrie* is dated 1442. ⟨…⟩ The age distribution of surviving complete or partial Latin *Liber-abbaci* manuscripts is not very different from that of the translations; 3 appear to be from the later 13th century, 4 from the 14th, 2 or 3 from the 15th, 3 or 2 from the 16th."

Hoyrup draws the following conclusions about the number of Fibonacci's works, present in the circulation, and the number of borrowings from them [22; p. 36]:

> "Obviously, all of these together count as almost nothing compared to the total number of abbacus manuscripts, and analysis of most treatises from the fourteenth and fifteenth centuries would reveal a picture similar to that of the Umbrian abbacus — with the difference that the number of borrowings from Fibonacci would be much smaller (mostly nil), …"

Pioneers of "Fibonacci numbers"

Was Fibonacci the pioneer of "Fibonacci numbers" or were they known before him? After all, people have long since been interested in different number sequences, which is what professor

Avadhesh Narayan Singh (1901−1954) notes in his article [47; p. 607]:

> "Series of numbers developing according to certain laws have attracted the attention of people in all times and climes. The Egyptians are known to have used the arithmetic series about 1550 B.C. Arithmetic as well as geometric series are found in the Vedic literature of the Hindus (c. 2000 B.C.)."

Some authors suggest, that the series, presented by Fibonacci, was known before him. Thus, in his article "The So-called Fibonacci Numbers in Ancient and Medieval India" Parmanand Singh writes [48; pp. 229–230]:

> "What are generally referred to as the Fibonacci numbers and the method for their formation were given by Virahāṅka (between A. D. 600 and 800), Gopāla (prior to A. D. 1135) and Hemacandra (C. A. D. 1150), all prior to L. Fibonacci (C. A. D. 1202). ⟨…⟩ Indian authorities on the metrical sciences used this sequence in works on metric."

In Fig. 9 we provide two tables with short (|) and long (*S*) vowel sounds[44], from the given article by Singh [48; pp. 231–232]: Table I, in which the number of morae in a meter is written above and its expansion below, and Table II, in which the expansion of the meter having 6 morae is given along with serial number of each variation, with 13 variations in total.

Based upon the arguments presented by him, Parmanand Singh concludes [48; p. 232]:

> "Here, it is easily seen that the variations of *mātrā- vṛttas*[45] form the sequence of numbers which are now called Fibonacci numbers. For, the numbers of variations of meters having 1, 2, 3, 4, 5, 6, … morae are, respectively, 1, 2, 3, 5, 8, 13, …, and these are the Fibonacci numbers. It is also ob-

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[44] In Sanskrit, an ancient language of India, same as in English, there are short and long vowels (syllables, sounds): 1 short syllable = 1 mora; 1 long syllable = 2 morae. Mora — the smallest unit of time in a verse.

[45] *Mātrā-vṛttas* — one of the categories of meters in Sanskrit and Prakrit poetry in India [50; c. 230]. — *Footnote ours.*

served that the method for finding the numbers of variations of *mātrā-vṛttas* leads to the general rule, $U_n = U_{n-1} + U_{n-2}$ for the formation of Fibonacci numbers. Thus it can be safely concluded that the concept of the sequence of these numbers in India is at least as old as the origin of the metrical sciences of Sanskrit and Prakrit[46] poetry."

TABLE I

1 mora	2 morae	3 morae	4 morae	5 morae
|	S	|S	SS	|SS
	||	S|	||S	S|S
		|||	|S|	|||S
			S|||	SS|
			||||	||S|
				|S||
				S|||
				|||||

TABLE II

(1) SSS	(6) |SS|	(10) ||S||
(2) ||SS	(7) S|S|	(11) |S|||
(3) |S|S	(8) |||S|	(12) S||||
(4) S||S	(9) SS||	(13) ||||||
(5) ||||S		

Fig. 9. The number of possible combinations of short (|) and long (*S*) syllables in a meter, containing from one to six morae

Donald Ervin Knuth (on his website he provides the pronunciation of his surname as "Ka-NOOTH") gives a references to this article by Singh in the first volume of his monumental work "The Art of Computer Programming" [25; p. 80]:

[46] Prakrit is general name for the ancient dialects of India. Prakrit languages were spoken in India between 600 BCE to 1000 CE.

"Before Fibonacci wrote his work, the sequence F_n had already been discussed by Indian scholars, who had long been interested in rhythmic patterns that are formed from one-beat and two-beat notes or syllables. The number of such rhythms having n beats altogether is F_{n+1}; therefore both Gopāla (before 1135) and Hemachandra (c. 1150) mentioned the numbers 1, 2, 3, 5, 8, 13, 21, … explicitly. [See P. Singh, *Historia Math.* **12** (1985), 229−244; …]"

Chapter VII

FIBONACCI'S CONTRIBUTION

or

Punishment of innocents and rewarding of non-participants

The question about Fibonacci's contribution is, perhaps, the most ambiguous. When you read about the Fibonacci's contribution to the treasury of world science, you are left with a feeling of certain incompleteness. Everyone knows that the contribution of Fibonacci is great, but nobody says how they know it. The great contribution — that is it! Apparently, it is believed if the man is great, so is his contribution. But high level of qualification of the member of society does not mean an automatic perception of his ideas. Recognition is not only the gratitude from the society, but also the use of the results of the author's works. Whereas Fibonacci, just after he wrote his works and received recognition from fellow citizens, disappeared and for 600 years he was practically not mentioned. His works, as became clear above, were almost never used and the number of their copies was negligible.

Let us arrange in chronological order what is known to us about Fibonacci and what is in one way or another associated with him.

400 years later

The year 1611 saw the publication of the small work "Strena Seu De Nive Sexangula " (*A New Year's Gift, or On the Six-Corne-*

red Snowflake) by Johannes Kepler [24]. This work was written in a somewhat humorous tone and contains the following passage:

> "Duo sunt corpora regularia, dodecahedron et icosahedron, quorum illud quinquangulis figuratur expresse, hoc triangularis quidem, sed in quinquanguli formam coaptatis. Utriusque horum corporum ipsiusque adeo quinquanguli structura perfici non potest sine proportione illa, quam hodierni geometrae divinam appellant. Est autem sic comparata, ut duo minores proportionis continuae termini iuncti constituant tertium; semperque additi duo proximi, constituant immediate sequentem, eadem semper durante proportione, in infinitum usque. In numeris exemplum perfectum dare est impossibile. Quo longius tamen progredimur ab unitate, hoc fit exemplum perfectius. Sint minimi 1 et 1 quos imaginaberis inaequales. Adde, fient 2; cui adde maiorem 1, fient 3; cui adde 2, fient 5; cui adde 3, fient 8; cui adde 5, fient 13; cui adde 8, fient 21. Semper enim ut 5 ad 8, sic 8 ad 13, fere; et ut 8 ad 13, sic 13 ad 21, fere." [24; p. 20]
>
> ("Of the two regular solids, the dodecahedron and the icosahedron, the former is made up precisely of pentagons, the latter of triangles but triangles that meet five at a point. Both of these solids, and indeed the structure of the pentagon itself, cannot be formed without the divine proportion [golden section] as modern geometers call it. It is arranged that the two lesser terms of a progressive series together constitute the third, and the two last, when added, make the immediately subsequent term and so on to infinity, as the same proportion continues unbroken. It is impossible to provide a perfect example in round numbers. However, the further we advance from the number one, the more perfect the example becomes. Let the smallest numbers be 1 and 1, which you must imagine as unequal. Add them, and the sum will be 2; add to this the greater of the 1's, result 3; add 2 to this, and get 5; add 3, get 8; 5 to 8, 13; 8 to 13, 21. As 5 is to 8, so 8 is to 13, approximately, and as 8 to 13, so 13 is to 21, approximately." [24; p. 21])

There is not a word about Leonardo in this work. Perhaps Kepler thought it unnecessary to mention Fibonacci. This does not seem likely however, since, as he continues, Kepler mentions many writers of antiquity and his contemporaries. He would probably also have mentioned Fibonacci, had he known about him.

From here, a few conclusions come to mind:

- Kepler did not know about Leonardo Pisano and his sequence [21; p. 162].
- Kepler knew about the existence of a whole class of sequences, formed by the rule: sum of two consecutive terms gives the third, and the sequence of type 1, 2, 3, 5, 8, … was only a special case.
- Kepler was aware of some properties of the sequences mentioned by him; he also knew about some numerical ratios inherent in plants. [24; p. 33]

As we can see, the mathematics continued to develop its own way without Leonardo and the sequence of type 1, 2, 3, 5, 8, … was likely discovered anew — perhaps for the *n*-th time in the history of mathematics.

600 years later

After the six centuries the situation relating to Fibonacci was as follows.

Gabriel Lamé

The year 1844

In the work "Note sur la limite du nombre des divisions dans la recherche du plus grand commun diviseur entre deux nombres entiers" (*Note on the limit of number of divisions in the search for greatest common divisor of two integers*), published in 1844, the famous French mathematician and engineer, professor Gabriel Lamé (1795−1870) writes [27; pp. 867−868]:

> "Si, commençant par 1 et 2, on compose une suite de nombres entiers, tels que chacun d'eux soit égal à la somme des deux nombres qui le précèdent, on obtient la série suivante:

(If, starting with 1 and 2, one composes a sequence of whole numbers, such that each one of them is equal to the sum of the two numbers which precede it, the following series is obtained:)

(I) 1, 2, 3, 5, 8; 13, 21, 34, 55, 89;

144, 233, 377, 610, 987; 1597 . . . . . ”

That is, Lamé is talking about Fibonacci's sequence, but does not mention Fibonacci's name. It seems he does not know about him.

Édouard Lucas

From below you can see how Edouard Lucas was "discovering" Fibonacci for himself.

The year 1873

In the work "Recherches sur l'analyse indéterminée et l'Arithmétique de Diophante" (*Research on indeterminate analysis and the Arithmetic of Diophantus*) Lucas for the first time makes the following observation [31; p. 450]:

> "Pourtant ce théorème a été énoncé antérieurement, mais démontré incomplètement, par Fibonacci (Léonard de Pise) dans son *Traité des nombres carrés* ; ce traité qu'on avait cru longtemps perdu a été retrouvé et publié par M. le prince Balthazar Boncompagni. (Voir *Journal de M. Liouville*, t. XX, p. 567.) ”
>
> (Though this theorem was previously provided by Fibonacci (Leonardo of Pisa) in his *Treatise of Square Numbers*, however it was not fully proved; this treatise, which was for a long time considered lost, was discovered and published by Mr. Prince Balthasar Boncompagni (See *Journal of Mr. Liouville*, t. XX, p. 567.))

The page 567 is not present in Liouville's volume XX. It appears that Lucas is referencing the pages 56 − 57 of the article by Woepcke "*Note sur le* Traité des nombres carrés, *de* Léonard de Pise, *retrouvé et publié par M. le prince* Balthasar Boncompagni," placed in this volume of Liouville's journal "Journal de mathématiques pures et appliquées, ou recueil mensuel de mémoires sur les diverses parties des mathématiques" for the year 1855 (publié par Joseph Liouville. Tome XX. – Anneé 1855. Paris).

From above it can be seen, that in 1873 Lucas already knew about Fibonacci.

The year 1876, January

In the journal "Comptes Rendus hebdomadaires des séances de l'Académie des sciences," the title of which roughly translates to the "Proceedings of the French Academy of Sciences," in the section for the January 10th there appears a short, two-and-a-half page note by Lucas "Note sur l'application des séries récurrentes à la recherche de la loi de distribution des nombres premiers" (*Note on the application of the recursion series to the research of the law of distribution of the prime numbers*) [32; pp. 165 −167].

The note begins as follows:

> "La série de Lamé, … est une série récurrente définie par la relation
>
> $$u_{n+2} = u_{n+1} + u_n,$$
>
> et par les deux conditions initiales
>
> $$u_0 = 0, \qquad u_1 = 1; \ldots\text{"}$$

Here Lucas talks about the Lame series as a recurring sequence (series), and from the provided formula and initial conditions it is clear, that he is talking about the Fibonacci sequence, although Lucas calls it the *series of Lame*. From this we can see that in January 1876 Lucas did not yet know about the Fibonacci's problem about rabbits.

The year 1876, May

Another work by Lucas — "Sur la théorie des nombres premiers" (*On the theory of prime numbers*) — appears [35], where on page 929 there is the following:

"… nous ferons observer que l'une de ces séries

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

donnée par la loi de récurrence

$$u_{n+2} = u_{n+1} + u_n,$$

et connue habituellement sous le nom de SÉRIE DE LAMÉ (1), a été définie pour la première fois par FIBONACCI (2)."

(… let us note, that one of these series 0, 1, 1, 2, 3, 5, 8, 13, 21, …, is given by the recurrence $u_{n+2} = u_{n+1} + u_n$, and more commonly known as the *series of Lame* (1), was first presented by Fibonacci (2).)

After this remark, later on in the article Lucas calls the specified series the series (sequence) of Fibonacci (série de Fibonacci). And when someone says that the sequence was given its name by Edouard Lucas in May 1876, apparently they mean this particular article. Koshy, for example, writes about it so [26; p. 5]:

"The sequence was given its name in May of 1876 by the outstanding French mathematician François-Edouard-Anatole-Lucas, … who had originally called it "the series of Lamé," after the French mathematician Gabriel Lamé (1795−1870)."

Footnote (2) in the article "Sur la théorie des nombres premiers" by Lucas, referenced here by us, is as follows:

"(2) *Il* Liber Abbaci *di Leonardo Pisano, pubblicato secondo la lezione del Codice Magliabechiano*, da B. BONCOMPAGNI, Roma, 1867, p. 284, — à la marge."

As we can see, Lucas learned about the sequences, presented by Fibonacci, from the published by Boncompagni Fibonacci's work *Liber Abaci* [9; vol. I].

The year 1877, March-May

After Fibonacci came to the attention of Lucas, there appears a big article "Recherches sur plusieurs ouvrages de Léonard de Pise et sur diverses questions d'arithmétique supérieure" [32] by

Lucas, directly addressing the works of Fibonacci. The introductory part of this article can be viewed as the call to the scientific community to pay attention to Fibonacci and his works and to recognize his priority in the problems set out by him.

In the same place, in the beginning of the article, Lucas provides in the original language (in Latin) the problem about rabbits, mentions a number of famous names of scientists (Claude Gaspard Bachet de Méziriac, d'Albert Girard, Robert Simson, Gabriel Lamé, Jacques Binet, Jean Plana) and notes [32; p. 9]:

> "… mais aucun des auteurs dont nous venons de parler, n'a attribué à FIBONACCI l'honneur de la découverte de cette série si remarquable."
>
> (… but none of the authors we have mentioned, have attributed to Fibonacci the honor of the discovery of this remarkable series.)

All of this suggests that neither Lucas (at that time), nor mentioned by him authors, as well as Kepler, did not know anything about the work of Fibonacci, containing this sequence, nor about the Fibonacci himself. Lucas himself, as we can see, learned about Fibonacci from the publications by Boncompagni.

Next, as in the May 1876 article, Lucas again talks about the authorship of Fibonacci in relation to the sequence, known under the name of Lame [32; p. 9]:

> "La série dite de Lamé, mais considérée pour la première fois par Léonard de Pise, ainsi que nous venons de le dire, est une série récurrente donnée par la relation
>
> $$u_{n+2} = u_{n+1} + u_n,$$
>
> et par les deux conditions initiales
>
> $$u_0 = 0, \ \ u_1 = 1."$$
>
> (The sequence, known under the name of Lame, but considered for the first time by Leonardo of Pisa, as we have just mentioned it, is a recurrent sequence, given by the relation
>
> $$u_{n+2} = u_{n+1} + u_n,$$

and two initial conditions

$$u_0 = 0 \text{ and } u_1 = 1.)$$

From the foregoing it is clear, why today we say "Fibonacci numbers," "Fibonacci sequence," "Fibonacci series."

Leonard Eugene Dickson

In Chapter XVII of the three-volume edition *History of the Theory of Numbers* a known American mathematician Leonard Eugene Dickson briefly lists events related to the study of Fibonacci's recurrent series and sequences, and also mentions a number of authors and the contribution of each [13]. From the provided data it follows that what is known to us about the properties of recurrent sequences was effectively acquired from the second half of the 17th century to the present. With regard to Fibonacci this means, that he had no part in it.

As a result, the following picture emerges:

1200

1202; 1228 — Fibonacci wrote *Liber Abaci*;
1288 − 1290 — the manuscript *Livero de l'abbecho* is written — borrowings from *Liber Abaci* are insignificant;
End of the 13th century — three copies of *Liber Abaci* are made.

1300

1307 — Jacopo wrote *Tractatus Algorismi* — borrowings from *Liber Abaci* are insignificant;
1350 — chapters 14 and 15 from *Liber Abaci* are translated;
In the 14th century four copies of *Liber Abaci* are made.

1400

1400 — another translation of the chapters 14 and 15 from *Liber Abaci* and translation of Fibonacci's work *Liber Quadratorum*;
1442 — Fibonacci's *Practica geometriae* is translated from Latin;

1464 — Master Benedetto wrote an Italian version of *Liber Quadratorum* [45; p. xvii];
1494 — Pacioli quoted extensively from *Liber quadratorum* in his *Summa* [6; p. 81], [45; p. xx];
In the 15th century there are two or three copies of *Liber Abaci* made.

1500

1506 — Perizolo mentions in his notes Leonardo already as "Fibonacci" and his book *Liber Abaci*;
In the 16th century there are three or two copies made of *Liber Abaci*.

1600

1611 — Johannes Kepler does not know about Fibonacci.

1700

1765 — Flaminio Dal Borgo knows of Fibonacci [5; p. 9];
1787 — Ranieri Tempesti — mentions Fibonacci [5; p. 10];
1790 — Giovanni Grimaldi knows of Fibonacci [5; p. 8];
1797 — Pietro Cossali is familiar with Fibonacci's works *Liber Abaci* and *Liber Quadratorum*.

1800

1806 — Girolamo Tiraboschi knows of Fibonacci [5; p. 9];
1812 — Giovanni Andres knows of Fibonacci [5; p. 10];
1812 — Giovanni Battista Guglielmini wrote "Elogio di Leonardo Pisano";
1815 — Nicollet knows of Fibonacci [5; p. 9];
1820 — John Leslie knows of Fibonacci [5; p. 8];
1837 — Michel Chasles knows of Fibonacci [5; p. 8];
1838 — Libri published chapter 15 from *Liber Abaci* [5; p. 8];
1844 — Gabriel Lame does know of Fibonacci and his sequence;
1846 — Gartz knows of Fibonacci [5; p. 9];
1847 — Augusto de Morgan knows of Fibonacci [5; p. 9];
1851−1857 — Boncompagni publishes the works of Fibonacci and everything he knows about Fibonacci;
1857 — Bonaini discovers the *pisan document*;

1867 — Milanesi found out that Leonardo's father's name was Guilielmus;
1873 — up to this year Edouard Lucas did not know of Fibonacci;
1876 — Edouard Lucas began to call the series of the type 1, 2, 3, 5, 8, … the *Fibonacci series*, and not *Lame*, as before, after he found this series in the work of Fibonacci *Liber Abaci*.

As we can see, this is few and far between in order for Fibonacci's works to be considered widely known — especially compared to the total number of works on mathematics, written, moreover, in the native language of the area. Hoyrup, for example, provides a table [22; p. 35], which shows that in the period from 1276 to 1500 there were 442 such works written. It is not surprising, that the thin thread of knowledge about the works of Fibonacci was broken. The works are discovered again in the late 18th − early 19th century and the interest in them and Fibonacci as a person appears from a purely historical point of view.

So the situation with the Fibonacci's contribution as a whole looks to be as follows: Fibonacci made a personal contribution, by writing his works, but society did not take advantage of them. In other words, Fibonacci deposited his works into the treasury of human knowledge, but mankind did not take them out of this treasury. The contribution did not take place and Fibonacci was thus unjustly punished by oblivion.

On the other hand, he was rewarded by becoming widely known in our time because of the unique properties of the sequence, mentioned by him in the problem about rabbits. However he had nothing to do with the discovery of these properties: it was not he who saw the remarkable properties of the provided by him sequence — it was done much later by others. He did not attach any importance at all to this problem. In other words, he received the reward in the form of wide popularity for something he had nothing to do with. It came out as in a cautionary tale — a kind of anecdote, reprinted in slightly different compositions in any number of project management books. In that, in any big project, with its inherent tendency toward failure, the last two phases of a project are: *Punishment of the innocent* and *Rewarding of non-participants*.

Thomas Koshy notices [26; p. 5]:

> “It is a bit ironic that despite Fibonacci’s numerous mathematical contributions, he is primarily remembered for this sequence that bears his name.”

Chapter VIII

WHY?

How could it happen that the works of Fibonacci, being in themselves virtually an encyclopedia on mathematics of that time, and their author were forgotten and unneeded?

Instead of an answer one can ask a counter question: “Who studies by encyclopedias?” Every learner wants the textbook to be clear and small in volume, so that there is nothing “extra” in it. Encyclopedias are usually used when the science is already mastered — they are referenced from time to time, and even then — not by everyone.

Although in the preface to *Liber Abaci* Fibonacci wrote, that the work is intended for the Latin people, it turned out too difficult for students, and for some mathematicians of the time as well, being in that sense ahead of its time. So one of the reasons, that prevented the spread of *Liber Abaci*, can be considered its universal, encyclopedic nature and its complexity: the common people did not need the advanced sections of the work, while mathematicians and scientists did not need applied problems, useful to traders and common people.

Next, let us compare, for example the works *Carmen de Algorismo*, *Algorismus* and *Liber Abaci* in terms of volume.

Carmen de Algorismo by Alexander de Villa Dei, in the collection *Rara Mathematica* under the title of “Alexandri de Villa Dei Carmen de Algorismo” [20; pp. 73−83], consists of 285 lines and takes up in there an incomplete 11 pages — together with the two large footnotes at the bottom of pages 73 and 74.

The treatise *Algorismus* by John Sacrobosco, in the same collection *Rara mathematica* under the title "Tractatus de arte numerandi" [20; pp. 1–26], takes up 26 pages of text.

Liber Abaci in Boncompagni's publication takes up 459 pages of text [9; vol. I], and in Sigler's book takes up 636 pages [44].

What did it mean in the times of Fibonacci to manually rewrite such a voluminous work as *Liber Abaci*? How much time had to be spent on rewriting? Where could one get that amount of writing material (parchment) and what would all this cost? As Alfred Lieber writes on the literacy of the medieval European traders [31; p. 239]:

> "The problem of the literacy of the medieval European merchant is bound up with the availability of reasonably priced writing materials — Italian paper-making started only at the very end of the thirteenth century …"

Liber Abaci thereby might have been unpopular due to the large amount of work involved in rewriting it (this reason was already stated previously) and the high cost of such an undertaking. Indeed, if a dozen or two pages were enough to learn arithmetic in University, and the lecturer, in addition, would explain the obscure places, then why rewrite hundreds of pages of *Liber Abaci*?

The third reason is the inaccessibility of *Liber Abaci*: if the copying of it was avoided, then where could one get a manuscript, if it existed in single copies? Interesting, was there a possibility of buying the work, and was anyone willing?

The fourth reason is the lack of necessity for this kind of literature in a household: the knowledge acquired fit into the head — thankfully not much was required for everyday life. There was no need to keep such a thick reference book handy.

CONCLUSION

The works of Fibonacci were, without a doubt, the works of the high scientific level, however they did not have a direct influence on the development of mathematical knowledge, were not known to the general and scientific community, were soon forgotten and only after centuries attracted attention solely from a historical point of view.

The role and importance of Fibonacci for the time, in which he lived, are practically unknown to us.

Today, only a narrow circle of people are acquainted with the works of Fibonacci, while he himself is effectively known only through the sequence of numbers, named after him, from the problem about rabbits. However he had nothing to do with the discovery of all those remarkable properties that are present in the sequence that bears his name. He himself did not single out the problem about rabbits among the others and did not talk about any unique properties of the obtained by him sequence of numbers. It was not Fibonacci, but others, who discovered them centuries later. If Edouard Lucas knew about the authors mentioned by Parmanand Singh, it is hard to say what the sequence would be called today. It makes little sense to advocate a name change — it became well established and sounds nice. And Fibonacci deserves it — even though not due to his works, but at least because of the sequence, named after him, he became widely known.

The Fibonacci's story has a happy ending: Fibonacci and his works are brought back from the obscurity, his merits and works are recognized and appreciated, his legacy is not lost and is made public.

Hard work and talent should be appreciated and recognized, even if their time has passed.

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As ancient Romans said:

Do ut des

(I give so that you may give!)

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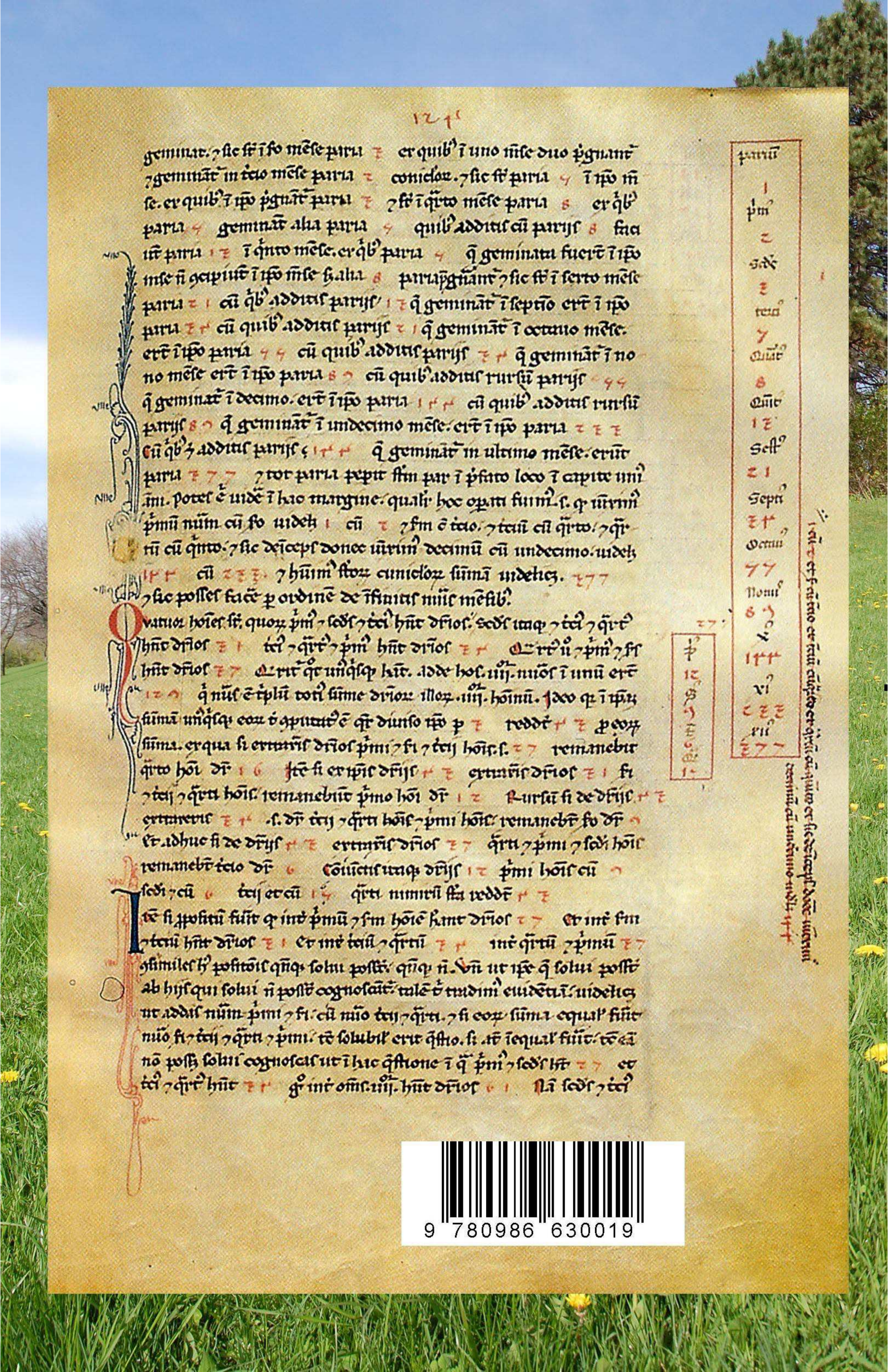

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